

A_2 -Planar Algebras II: Planar Modules

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Abstract

Generalizing Jones's notion of a planar algebra, we have previously introduced an A_2 -planar algebra capturing the structure contained in the $SU(3)$ \mathcal{ADE} subfactors. We now introduce the notion of modules over an A_2 -planar algebra, and describe certain irreducible Hilbert A_2 - TL -modules. A partial decomposition of the A_2 -planar algebras for the \mathcal{ADE} nimrep graphs associated to $SU(3)$ modular invariants is achieved.

1 Introduction

We introduced in [7] the notion of an A_2 -planar algebra. This was useful to understand the double complexes of finite dimensional algebras which arise in the context of $SU(3)$ subfactors and modular invariants. Here we begin a study of their planar modules.

These planar algebras are direct generalization of the planar algebras of Jones [15]. To avoid too much confusion one could refer to these planar algebras of Jones here as A_1 -planar algebras which naturally contain the Temperley-Lieb algebra which encodes the representation theory of quantum $SU(2)$.

Our A_2 -planar algebras naturally encode the representation theory of quantum $SU(3)$ or in the dual Hecke picture, the finite dimensional algebras which appear from the representations of the deformation of the symmetric group with generators the self-adjoint operators $1, U_1, U_2, \dots, U_{n-1}$ and relations:

$$\begin{aligned} \text{H1:} \quad & U_i^2 = \delta U_i, \\ \text{H2:} \quad & U_i U_j = U_j U_i, \quad |i - j| > 1, \\ \text{H3:} \quad & U_i U_{i+1} U_i - U_i = U_{i+1} U_i U_{i+1} - U_{i+1}, \end{aligned}$$

where $\delta = q + q^{-1}$, $q \in \mathbb{C}$, and the extra relation

$$(U_i - U_{i+2}U_{i+1}U_i + U_{i+1})(U_{i+1}U_{i+2}U_{i+1} - U_{i+1}) = 0. \quad (1)$$

In [5] we computed the numerical values of the Ocneanu cells, announced by Ocneanu (e.g. [23, 24]), and consequently representations of the Hecke algebra, for the $SU(3)$ \mathcal{ADE} graphs. These cells give numerical weight to Kuperberg's [20] diagram of trivalent vertices – corresponding to the trivial representation is contained in the triple product of the fundamental representation of $SU(3)$ through the determinant. They will yield in a natural way, representations of an A_2 -Temperley-Lieb or Hecke algebra. For $SU(2)$ or bipartite graphs, the corresponding weights (associated to the diagrams of cups or caps), arise in a more straightforward fashion from a Perron-Frobenius eigenvector, giving a natural representation of the Temperley-Lieb algebra or Hecke algebra.

The bipartite theory of the $SU(2)$ setting has to some degree become a three-colourable theory in our $SU(3)$ setting. This theory is not completely three-colourable since some of the graphs are not three-colourable – namely the graphs $\mathcal{A}^{(n)*}$, $n \geq 4$, and $\mathcal{D}^{(n)}$, $n \not\equiv 0 \pmod{3}$. The figures for the complete list of the \mathcal{ADE} graphs are given in [1, 5].

2 Preliminaries on Jones' planar algebras and planar modules

Let us briefly review the basic construction of Jones' A_1 -planar algebras, and the notion of planar modules over these algebras. A planar k -tangle consists of a disc D in the plane with $2k$ vertices on its boundary, $k \geq 0$, and $n \geq 0$ internal discs D_j , $j = 1, \dots, n$, where the disc D_j has $2k_j$ vertices on its boundary, $k_j \geq 0$. One vertex on the boundary of each disc (including the outer disc D) is chosen as a marked vertex, and the segment of the boundary of each disc between the marked vertex and the vertex immediately adjacent to it as we move around the boundary in an anti-clockwise direction is labelled $*$. Inside D we have a collection of disjoint smooth curves, called strings, where any string is either a closed loop, or else has as its endpoints the vertices on the discs, and such that every vertex is the endpoint of exactly one string. Any tangle must also allow a checkerboard colouring of the regions inside D .

The planar operad is the collection of all diffeomorphism classes of such planar tangles, with composition of planar tangles defined. A planar algebra P is then defined to be an algebra over this operad, i.e. a family $P = (P_0^+, P_0^-, P_k, k > 0)$ of vector spaces with $P_0^\pm \subset P_k \subset P_{k'}$ for $0 < k < k'$, and such that for every k -tangle T with n internal discs D_j labelled by elements $x_j \in P_{k_j}$, $j = 1, \dots, n$, there is an associated linear map $Z(T) : \bigotimes_{j=1}^n P_{k_j} \rightarrow P_k$, which is compatible with the composition of tangles and re-ordering of internal discs.

A planar module over P is a graded vector space $V = (V_0^+, V_0^-, V_k, k > 0)$ with an action of P . Given a planar m -tangle T in \mathcal{P} with distinguished (“ V input”) internal disc D_1 with $2k$ vertices on its boundary, $k \geq 0$, and other (“ P input”) internal discs D_p , $p = 2, \dots, n$, with $2k_p$ vertices on its boundary, $k_p \geq 0$, there is a linear map $Z(T) : V_k \otimes (\bigotimes_{p=2}^n P_{k_p}) \rightarrow V_m$, where $Z(T)$ satisfies the same compatibility conditions as for P itself.

2.1 $P^{\mathcal{G}}$ as a TL -module for an ADE Dynkin diagram \mathcal{G}

In the case of $SU(2)$, Jones [14] determined all Hilbert modules $H^{k,\omega}$ of lowest weight $k > 0$, $k \in \mathbb{N}$, and H^0 of lowest weight 0. We will give a brief overview of these modules. For $k, m \in \mathbb{N}$, let $ATL_{m,k}$ denote the space of all annular (m, k) -tangles (having $2m$ vertices on the outer disc and $2k$ vertices on the (distinguished) inner disc, where the vertices have alternating orientations) with no other internal discs, where composition of tangles is defined by inserting one annular (m, k) -tangle inside the internal disc of an annular (k, n) -tangle. For $1 \leq k \leq m$, $k, m \in \mathbb{N}$, let \mathcal{T}_m^k denote the set of annular (m, k) -tangles with no internal discs and $2k$ through strings. If $\widetilde{ATL}_{m,k}$ denotes the quotient of $ATL_{m,k}$ by the ideal generated by all annular (m, k) -tangles with no internal discs and strictly less than $2k$ through strings, then the equivalence classes of the elements of \mathcal{T}_m^k form a basis for $\widetilde{ATL}_{m,k}$. The group \mathbb{Z}_k acts by an internal rotation, which permutes the basis elements. The action of ATL on $\widetilde{ATL}_{m,k}$ is given as follows. Let T be an annular (p, m) -tangle in $ATL_{p,m}$ and $R \in \mathcal{T}_m^k$. Define $T(R)$ to be $\delta^r \widehat{TR}$ if the (p, k) -tangle TR has $2k$ through strings and 0 otherwise, where TR contains r contractible circles and \widehat{TR} is the tangle TR with all the contractible circles removed. Since the action of ATL commutes with the action of \mathbb{Z}_k , as a TL -module $\widetilde{ATL}_{m,k}$ splits as a direct sum, over the k^{th} roots of unity ω , of TL -modules $V_m^{k,\omega}$ which are the eigenspaces for the action of \mathbb{Z}_k with eigenvalue ω . For each k one can choose a faithful trace tr on the abelian C^* -algebra $\widetilde{ATL}_{k,k}$, which extends to $ATL_{k,k}$ by composition with the quotient map. The inner-product on $\widetilde{ATL}_{m,k}$ is then defined to be $\langle S, T \rangle = \text{tr}(T^*S)$ for $S, T \in \widetilde{ATL}_{m,k}$.

We now turn to the zero-weight case ($k = 0$). The algebras ATL_{\pm} , which have the regions adjacent to both inner and outer boundaries shaded \pm , are generated by elements $\sigma_{\pm}\sigma_{\mp}$, where σ_{\pm} is the (\pm, \mp) -tangle which is just a single non-contractible circle, with the region which meets the outer boundary shaded \pm and the region which meets the inner boundary shaded \mp . Then the dimensions on V_+ and V_- must be 1 or 0 for any TL -module V . Then in V , the maps $\sigma_{\pm}\sigma_{\mp}$ must contribute a scalar factor μ^2 , where $0 \leq \mu \leq \delta$. If $\mu = \delta$, V^{δ} is simply the ordinary Temperley-Lieb algebra. When $0 < \mu < \delta$, V^{μ} is the TL -module such that V_m^{μ} has as basis the set of $(m, +)$ -tangles with no internal discs and at most one non-contractible circle. The action of ATL on V^{μ} , $0 \leq \mu \leq \delta$, is given as follows. Let T be an annular (p, m) -tangle in $ATL_{p,m}$ and R be a basis element of V^{μ} . Define $T(R)$ to be $\delta^r \mu^{2d} \widehat{TR}$, where TR contains r contractible circles and $2d + i$ non-contractible circles, where $i \in \{0, 1\}$, and \widehat{TR} is the tangle TR with all the contractible circles removed and $2d$ of the non-contractible circles removed. The inner product on V^{μ} is defined by $\langle S, T \rangle = \delta^r \mu^{2d}$, where T^*S contains r contractible circles and $2d$ non-contractible circles. When $\mu = 0$, we have TL -modules $V^{0,+}$ and $V^{0,-}$, where $V_m^{0,\pm}$ has as basis the set of (m, \pm) -tangles with no internal discs and no contractible circles. The action of ATL on $V^{0,\pm}$ is given as follows. Let T be an annular (p, m) -tangle in $ATL_{p,m}$ and R be a basis element of $V^{0,\pm}$. Define $T(R)$ to be $\delta^r \widehat{TR}$, where TR contains r contractible circles. Now \widehat{TR} is zero if TR contains any non-contractible circles, and is the tangle TR with all the contractible circles removed otherwise. The inner product on $V^{0,\pm}$ is defined by $\langle S, T \rangle = 0$ if T^*S contains any non-contractible circles, and $\langle S, T \rangle = \delta^r$ otherwise, where r is the number of contractible circles in T^*S .

In the generic case, $\delta > 2$, it was shown that the inner-product is always positive

definite, so that $H = V$ is a Hilbert TL -module, for the irreducible lowest weight k TL -module V . In the non-generic case, if the inner product is positive semi-definite, H is defined to be the quotient of V by the vectors of zero-length with respect to the inner product.

Let \mathcal{G} be a bipartite graph. Then the vertex set of \mathcal{G} is given by $\mathfrak{V} = \mathfrak{V}_+ \cup \mathfrak{V}_-$, where there are no connecting a vertex in \mathfrak{V}_+ to another, and similarly for \mathfrak{V}_- . We call the vertices in \mathfrak{V}_+ , \mathfrak{V}_- the even, odd respectively vertices of \mathcal{G} , and the distinguished vertex $*$ of \mathcal{G} , which has the highest Perron-Frobenius weight, is an even vertex. The adjacency matrix of \mathcal{G} can thus be written in the form $\begin{pmatrix} 0 & \Lambda_{\mathcal{G}} \\ \Lambda_{\mathcal{G}}^T & 0 \end{pmatrix}$. We let $r_{\pm} = |\mathfrak{V}_{\pm}|$. The planar algebra $P^{\mathcal{G}}$ of a bipartite graph \mathcal{G} was constructed in [13], which is the path algebra on \mathcal{G} where paths may start at any of the even vertices of \mathcal{G} , and where the m^{th} graded part $P_m^{\mathcal{G}}$ is given by all pairs of paths of length m on \mathcal{G} which start at the same even vertex and have the same end vertex. Let μ_j , $j = 1, \dots, r_+$, denote the eigenvalues of $\Lambda_{\mathcal{G}} \Lambda_{\mathcal{G}}^T$. Then the following result is given in [26, Prop. 13], which motivated Proposition 5.4: The irreducible weight-zero submodules of $P^{\mathcal{G}}$ are H^{μ_j} , $j = 1, \dots, r_-$, and $r_+ - r_-$ copies of H^0 , and these can be assumed to be mutually orthogonal.

Reznikoff [26] computed the decomposition of $P^{\mathcal{G}}$ as a TL -module into irreducible TL -modules for the ADE Dynkin diagrams. For the graphs A_m , $m \geq 3$,

$$P^{A_m} = \bigoplus_{j=1}^s H^{\mu_j}, \quad (2)$$

where $s = \lfloor (m+1)/2 \rfloor$ is the number of even vertices of A_m and $\mu_j = 2 \cos(j\pi/(m+1))$, $j = 1, \dots, s$. For D_m , $m \geq 3$,

$$P^{D_m} = \bigoplus_{j=1}^t H^{\mu_j} \oplus (s-t)H^{0,\pm} \oplus \bigoplus_{j=1}^{s-2} H^{2j,-1}, \quad (3)$$

where $s = \lfloor (m+2)/2 \rfloor$, $t = \lfloor (m-1)/2 \rfloor$ are the number of even, odd vertices respectively of D_m , and $\mu_j = 2 \cos((2j-1)\pi/(2m-2))$, $j = 1, \dots, t$. For the exceptional graphs the results are

$$P^{E_6} = H^{\mu_1} \oplus H^{\mu_4} \oplus H^{\mu_5} \oplus H^{2,-1} \oplus H^{3,\omega} \oplus H^{3,\omega^{-1}}, \quad (4)$$

$$P^{E_7} = H^{0,\pm} \oplus H^{\mu_1} \oplus H^{\mu_5} \oplus H^{\mu_7} \oplus H^{2,-1} \oplus H^{3,\omega} \oplus H^{3,\omega^{-1}} \oplus H^{4,-1} \oplus H^{8,-1}, \quad (5)$$

$$P^{E_8} = H^{\mu_1} \oplus H^{\mu_7} \oplus H^{\mu_{11}} \oplus H^{\mu_{13}} \oplus H^{2,-1} \oplus H^{3,\omega} \oplus H^{3,\omega^{-1}} \oplus H^{4,-1} \\ \oplus H^{5,\zeta} \oplus H^{5,\zeta^{-1}} \oplus H^{5,\zeta^2} \oplus H^{5,\zeta^{-2}}, \quad (6)$$

where $\omega = e^{2\pi i/3}$, $\zeta = e^{2\pi i/5}$, and $\mu_j = 2 \cos(\pi j/h)$ where h is the Coxeter number.

3 A_2 -Planar Algebras

We will now review the basic construction of our A_2 -planar algebras. An A_2 -planar i, j -tangle consists of a disc $D = D_0$ in the plane together with a finite (possibly empty) set of disjoint sub-discs D_1, D_2, \dots, D_n in the interior of D . Each disc D_k , $k \geq 0$, will have




Figure 1: Trivalent vertices

an even number $2(i_k + j_k) \geq 0$ of vertices on its boundary ∂D_k ($i_0 = i, j_0 = j$). A vertex will be called a source vertex if the string attached to it has orientation away from the vertex and a sink vertex if the string attached has orientation towards the vertex. The first j_k vertices are restricted to be sources, the next $2i_k$ vertices alternate between sources and sinks (with vertex $j_k + 1$ a source), and finally the last j_k vertices are all sinks. We will use the convention of numbering the vertices along the bottom edge (the last $i_k + j_k$ vertices) by $1, \dots, i_k + j_k$ in reverse order, so that the $2(i_k + j_k)$ -th vertex is called the first vertex along the bottom edge. We say that D_k has pattern i_k, j_k . Inside D we have a tangle where the endpoint of any string is either a trivalent vertex (see Figure 1) or one of the vertices on the boundary of a disc D_k , $k = 0, \dots, n$, or else the string forms a closed loop. Each vertex on the boundaries of the D_k is the endpoint of exactly one string, which meets ∂D_k transversally.

The regions inside D have as boundaries segments of the ∂D_k or the strings. These regions are labelled $\bar{0}$, $\bar{1}$ or $\bar{2}$, called the colouring, such that if we pass from a region R of colour \bar{a} to an adjacent region R' by passing to the right over a vertical string with downwards orientation, then R' has colour $\overline{a+1} \pmod{3}$. The segment of each ∂D_k between the last and first vertices is marked with $*_{b_k}$, $b_k \in \{0, 1, 2\}$, so that the region inside D which meets ∂D_k at this segment is of colour \bar{b}_k , and the choice of these $*_{b_k}$ must give a consistent colouring of the regions. For the outer boundary ∂D we impose the restriction $b_0 = 0$. For $i, j = 0, 0$ we have three types of tangle, depending on the colour \bar{b} of the region near ∂D .

An A_2 -planar i, j -tangle T with an internal disc D_l with $i_l, j_l = i', j'$ vertices on its boundary can be composed with another A_2 -planar i', j' -tangle S , which has external disc D' such that the orientations of the vertices on its boundary are consistent with those of D_l , giving a new i, j -tangle $T \circ_l S$, by inserting the A_2 -tangle S inside the inner disc D_l of T such that the vertices on the outer disc of S coincide with those on the disc D_l and the regions marked by $*$ also coincide. The boundary of the disc D_l is then removed, and the strings are smoothed if necessary. We let $\tilde{\mathcal{P}}$ be the collection of all diffeomorphism classes of such A_2 -planar tangles, with composition defined as above. The A_2 -planar operad \mathcal{P} is the quotient of $\tilde{\mathcal{P}}$ by the Kuperberg relations K1-K3 below, which are relations on a local part of the diagram:

$$\begin{array}{ll}
 \text{K1:} & \begin{array}{c} \text{A circle with a counter-clockwise arrow} \end{array} = \alpha \\
 \text{K2:} & \begin{array}{c} \text{A vertical line with a loop on the left side, containing two arrows: one pointing up and one pointing down} \end{array} = \delta \begin{array}{c} \text{A vertical line with a downward arrow} \end{array}
 \end{array}$$

K3: 

where $\delta = [2]_q$, $\alpha = [3]_q = \delta^2 - 1$, and the quantum number $[m]_q$ is defined by $[m]_q = (q^m - q^{-m})/(q - q^{-1})$, for some variable $q \in \mathbb{C}$.

An A_2 -planar algebra is then defined to be an algebra over this operad, i.e. a family

$$P = (P_{0,0}^{\bar{a}}, a \in \{0, 1, 2\}, P_{i,j}, i, j > 0, i, j \neq 0, 0)$$

of vector spaces with $P_{0,0}^{\bar{a}} \subset P_{i,j} \subset P_{i',j'}$ for $0 < i \leq i'$, $0 < j \leq j'$, $a \in \{0, 1, 2\}$, and with the following property: for every labelled i, j -tangle $T \in \mathcal{P}_{i,j}$ with internal discs D_1, D_2, \dots, D_n , where D_k has pattern i_k, j_k , there is associated a linear map $Z(T) : \bigotimes_{k=1}^n P_{i_k, j_k} \longrightarrow P_{i,j}$ which is compatible with the composition of tangles in the following way. If S is a i_k, j_k -tangle with internal discs D_{n+1}, \dots, D_{n+m} , where D_k has pattern i_k, j_k , then the composite tangle $T \circ_l S$ is a i, j -tangle with $n + m - 1$ internal discs D_k , $k = 1, 2, \dots, l-1, l+1, l+2, \dots, n+m$. From the definition of an operad, associativity means that the following diagram commutes:

$$\begin{array}{ccc} \left(\bigotimes_{\substack{k=1 \\ k \neq l}}^n P_{i_k, j_k} \right) \otimes \left(\bigotimes_{k=n+1}^{n+m} P_{i_k, j_k} \right) & \searrow^{Z(T \circ_l S)} & \\ \text{id} \otimes Z(S) \downarrow & & P_{i,j}, \\ \bigotimes_{k=1}^n P_{i_k, j_k} & \nearrow_{Z(T)} & \end{array} \quad (7)$$

so that $Z(T \circ_l S) = Z(T')$, where T' is the tangle T with $Z(S)$ used as the label for disc D_l . We also require $Z(T)$ to be independent of the ordering of the internal discs, that is, independent of the order in which we insert the labels into the discs. If $i = j = 0$, we adopt the convention that the empty tensor product is the complex numbers \mathbb{C} .

4 A_2 -Planar Modules and A_2 -ATL

We now extend Jones's notion of planar algebra modules and the annular Temperley algebra to our A_2 -planar algebras (cf. Section 2 of [14]).

Definition 4.1 (cf. [14, Def. 2.1]) An A_2 -**annular tangle** T will be a tangle in \mathcal{P} with the choice of a distinguished internal disc, which we will call the inner disc. In particular, T will be called an A_2 -**annular** $(m_1, m_2 : k_1, k_2)$ -**tangle** if it is an A_2 -annular tangle with pattern m_1, m_2 on its outer disc and pattern k_1, k_2 on its inner disc. If $m_1 = m_2 = 0$ or $k_1 = k_2 = 0$, we replace the 0, 0 with \bar{a} , $a \in \{0, 1, 2\}$, corresponding to the colour of the region which meets the outer or inner disc respectively. When $m_1 = k_1$ and $m_2 = k_2$ we will call T an A_2 -annular m_1, m_2 -tangle.

Note, this annular tangle is different to those defined in [7]- here more than one internal disc is allowed, but one of those is chosen to be the distinguished disc (the inner boundary of the "annulus").

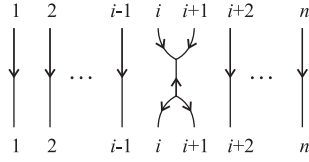


Figure 2: The m -tangle W_i , $i = 1, \dots, m-1$.

Definition 4.2 (cf. [14, Def. 2.2]) If P is an A_2 -planar algebra, a **module** over P , or P -**module**, will be a graded vector space $V = (V_{i,j}, i, j \geq 0, i, j \neq 0, 0, V_{0,0}^{\bar{a}})$ with an action of P . Given an A_2 -annular $(i, j : i', j')$ -tangle T in \mathcal{P} with distinguished (“ V input”) internal disc D_1 with pattern i', j' and other (“ P input”) internal discs D_p , $p = 2, \dots, n$, with patterns i_p, j_p , there is a linear map $Z(T) : V_{i',j'} \otimes (\otimes_{p=2}^n P_{i_p,j_p}) \rightarrow V_{i,j}$. The map $Z(T)$ satisfies the same compatability condition (7) for the composition of tangles as P itself.

An A_2 -planar algebra is always a module over itself- we will call it the **trivial module**. Any relation (i.e. linear combination of labelled A_2 -planar tangles) that holds in P will hold in V , e.g. K1-K3 hold in V where α, δ have the same values as in P .

A module over an A_2 -planar algebra P can be understood as a module over the A_2 -annular algebra A_2-AP , defined as follows. We define the associated annular category A_2-AnnP to have three objects \bar{a} for $i = j = 0$, $a \in \{0, 1, 2\}$, and one object for each $i, j \geq 0$ with i, j not both equal to zero, and whose morphisms are A_2 -annular labelled tangles with labelling set all of P . Let A_2-FAP be the linearization of A_2-AnnP - it has the same objects, but the set of morphisms from object i, j to object i', j' is the vector space having as basis the morphisms in A_2-AnnP from i, j to i', j' . Composition of morphisms in A_2-FAP is by linear extension of composition in A_2-AnnP . The **A_2 -annular algebra** $A_2-AP = \{A_2-AP(i, j : i', j')\}$ is the quotient of A_2-FAP by relations K1-K3.

Definition 4.3 (cf. [14, Def. 2.6]) We define $A_2-AP_{i,j}$ to be the algebra $A_2-AP(i, j : i, j)$ for $i, j \geq 0$ with i and j not both zero, and $A_2-AP_{\bar{a}}$, $a \in \{0, 1, 2\}$, to be the algebras spanned by A_2 -annular tangles with no vertices on the outer and inner boundaries, and with the regions which meet the boundaries coloured \bar{a} .

We apply this procedure to the A_2 -Temperley-Lieb algebra $\mathcal{V}^{A_2} = \text{alg}(1, W_i, i \geq 1)$ for fixed $\delta \in \mathbb{C}$, where W_i are the tangles illustrated in Figure 2. This algebra was shown in [7] to be isomorphic to the A_2 -Temperley-Lieb algebra. The labels for the internal discs are now just A_2 -annular tangles. For $m_1, m_2, n_1, n_2 \geq 0$ let $A_2-AnnTL(m_1, m_2 : n_1, n_2)$ be the set of all basis A_2 -annular $(m_1, m_2 : n_1, n_2)$ -tangles. Elements of $A_2-AnnTL(m_1, m_2 : n_1, n_2)$ define elements of $A_2-ATL(m_1, m_2 : n_1, n_2)$ by passing to the quotient of A_2-FATL by relations K1-K3. The objects of A_2-ATL are $\bar{0}$, $\bar{1}$ and $\bar{2}$ for $m_1 = m_2 = 0$. When m_1 and m_2 are not both equal to zero, the objects are the sets of $2(m_1 + m_2)$ points with pattern m_1, m_2 . $A_2-ATL_{m_1, m_2}(\delta)$ has as basis the set of A_2 -annular m_1, m_2 -tangles with no contractible circles, or embedded circles or squares. However, non-contractible circles are allowed, which make each algebra $A_2-ATL_{m_1, m_2}$ infinite dimensional. Multiplication in $A_2-ATL_{m_1, m_2}(\delta)$ is by composition of tangles, then reducing the resulting tangle using

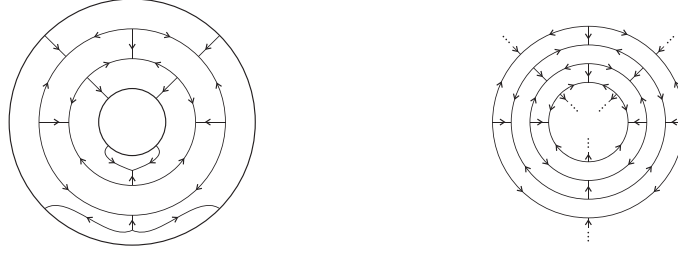


Figure 3: A basis A_2 -annular $(2, 0 : 2, 0)$ -tangle containing hexagons, and the possibility of an infinite number of hexagons

relations K1-K3 to remove closed loops (K1), embedded circles (K2) or embedded squares (K3).

For all $m_1, m_2 \geq 0$ such that $m_1 + m_2 \geq 2$, the algebras $A_2\text{-}ATL_{m_1, m_2}$ are also infinite dimensional due to the possibility of an infinite number of embedded hexagons in basis tangles in the annular picture, as illustrated in Figure 3.

We have a notion of the rank of a tangle. A minimal cut loop γ in an annular $(i, j : i', j')$ -tangle T will be a clockwise closed path which encloses the distinguished internal disc and crosses the least number of strings. We associate a weight $w_\gamma = (t_1, t_2)$ to a minimal cut loop γ , where t_1 is the number of strings of T that cross γ with orientation from left to right, and t_2 the number of strings that have orientation from right to left, as we move along γ in a complete clockwise loop. For a weight (t_1, t_2) , let $t_{\max} = \max\{t_1, t_2\}$ and $t_{\min} = \min\{t_1, t_2\}$. We will say (t'_1, t'_2) is less than (t_1, t_2) , and write $(t'_1, t'_2) < (t_1, t_2)$, if $t'_1 + t'_2 < t_1 + t_2$, and if $t'_1 + t'_2 = t_1 + t_2$ then $(t'_1, t'_2) < (t_1, t_2)$ if $2t'_{\max} + t'_{\min} < 2t_{\max} + t_{\min}$. The **rank** of T is then given by the smallest weight $w_\gamma = (t_1, t_2)$ associated to a minimal cut loop, such that $(t_1, t_2) \leq w_{\gamma'}$ for all other minimal cut loops γ' .

Let $A_2\text{-}AnnTL(m_1, m_2 : m_1, m_2)_{(t_1, t_2)}$ denote the set of tangles in $A_2\text{-}AnnTL(m_1, m_2 : m_1, m_2)$ with rank (t_1, t_2) . Since the rank cannot increase under composition of tangles, the linear span of $A_2\text{-}AnnTL(m_1, m_2 : m_1, m_2)_{(t_1, t_2)}$ for all $(t_1, t_2) < (t'_1, t'_2)$ for any fixed t'_1, t'_2 is an ideal in $A_2\text{-}ATL_{m_1, m_2}$.

Lemma 4.4 (cf. [14, Lemma 2.10]) *Let P be an A_2 -planar algebra and let t_1, t_2 satisfy $2t_{\max} + t_{\min} = 3m$. For any t'_1, t'_2 such that $2t'_{\max} + t'_{\min} \leq 3m$, denote by $A_2\text{-}AP_{t_1, t_2}^{(t'_1, t'_2)}$ the linear span in the algebra $A_2\text{-}AP_{t_1, t_2}$ of all labelled A_2 -annular t_1, t_2 -tangles with rank $(s_1, s_2) < (t'_1, t'_2)$. Then $A_2\text{-}AP_{(t_1, t_2)}^{(t'_1, t'_2)}$ is a two-sided ideal.*

Remark: For $A_2\text{-}ATL$ the quotient of $A_2\text{-}ATL_{t_1, t_2}$ by the ideal $A_2\text{-}AP_{t_1, t_2}^{(t'_1, t'_2)}$ is not in general finite dimensional, for $2t'_{\max} + t'_{\min} \leq 2t_{\max} + t_{\min}$. For example, consider the quotient of $A_2\text{-}ATL_{t_1, t_2}$ by $A_2\text{-}AP_{t_1, t_2}^{(3k, 0)}$ (or $A_2\text{-}AP_{t_1, t_2}^{(0, 3k)}$), for $3 \leq 3k \leq 2t_{\max} + t_{\min}$. The elements $\varphi_{(3k, 0)}$ and $\varphi_{(0, 3k)}$ (see Figure 4) have ranks $(3k, 0)$ and $(0, 3k)$ respectively, and can be composed an infinite number of times, but the resulting tangle cannot be reduced using K1-K3.

Lemma 4.5 (cf. [14, Lemma 2.11]) *Let $V = (V_{i, j})$ be a P -module. Then V is indecomposable if and only if $V_{i, j}$ is an indecomposable $A_2\text{-}AP_{i, j}$ -module for each $i, j \geq 0$.*



Figure 4: $\varphi_{(3k,0)}$ and $\varphi_{(0,3k)}$

Definition 4.6 (cf. [14, Def. 2.12]) The **weight** $wt(V)$ of a P -module V is the smallest integer $i + j$ for which $V_{i,j}$ is non-zero. If $V_{\bar{a}}$ is non-zero for $a \in \{0, 1, 2\}$ we say V has weight zero. Elements of $V_{i',j'}$ for $i' + j' = wt(V)$ will be called lowest weight vectors in V , and $V_{i',j'}$ is an A_2 - $AP_{i',j'}$ -module which we call a **lowest weight module**.

Note that for $i' + j' = wt(V)$, all $V_{i'+k,j'-k}$, $-i' \leq k \leq j'$, are lowest weight modules for V .

Definition 4.7 (cf. [14, Def. 2.13]) The Hilbert series (called the dimension in [14]) of a P -module V is the formal power series

$$\Phi_V(z_1, z_2) = \frac{1}{3} \dim(V_{\bar{0}} \oplus V_{\bar{1}} \oplus V_{\bar{2}}) + \sum_{\substack{i,j=0 \\ i,j \text{ not both } =0}}^{\infty} \dim(V_{i,j}) z_1^i z_2^j.$$

4.1 Hilbert P -modules

If P is a C^* - A_2 -planar algebra, the $*$ -algebra structure on P induces a $*$ -structure on A_2 - AP , where the involution $*$ is defined by reflecting an A_2 -annular $(m_1, m_2 : k_1, k_2)$ -tangle T about a circle halfway between the inner and outer disc, and reversing the orientation. T^* will be an A_2 -annular $(k_1, k_2 : m_1, m_2)$ -tangle. If P is a C^* - A_2 -planar algebra this defines an antilinear involution $*$ on A_2 - FAP by taking the $*$ of the underlying unlabelled tangle for a labelled tangle T , replacing the labels of T by their $*$'s, and extending by antilinearity. Since P is an A_2 -planar $*$ -algebra, all the A_2 -planar relations are preserved under $*$ on A_2 - FAP , so $*$ passes to an antilinear involution on the algebra A_2 - AP . In particular, all the A_2 - $AP_{i,j}$ are $*$ -algebras.

Definition 4.8 (cf. [14, Def. 3.1]) Let P be a C^* - A_2 -planar algebra. A P -module H will be called a **Hilbert P -module** if each $H_{i,j}$ is a finite dimensional Hilbert space with inner-product $\langle \cdot, \cdot \rangle$ satisfying

$$\langle av, w \rangle = \langle v, a^* w \rangle, \quad (8)$$

for all $v, w, \in H$ and $a \in A_2$ - AP .

As in the $SU(2)$ situation, a P -submodule of a Hilbert P -module is a Hilbert P -module. Also, the orthogonal complement of a P -submodule is a P -module, so that indecomposability and irreducibility are the same for Hilbert P -modules. The following Lemmas are $SU(3)$ versions of Lemmas in [14], the proofs of which are very similar to those in the $SU(2)$ situation.

Lemma 4.9 (cf. [14, Lemma 3.4]) *Let P be an A_2 - C^* -planar algebra and H a Hilbert P -module. If $W \subseteq H_{i,j}$ is an irreducible A_2 - $AP_{i,j}$ -submodule of $H_{i,j}$ for some i, j , then A_2 - $AP(W)$ is an irreducible P -submodule of H .*

Lemma 4.10 (cf. [14, Lemma 3.5]) *Let P be an A_2 - C^* -planar algebra and H a Hilbert P -module. Let V and W be orthogonal A_2 - $AP_{i,j}$ invariant subspaces of $H_{i,j}$ for some i, j . Then A_2 - $AP(V)$ is orthogonal to A_2 - $AP(W)$.*

As in the proof of [7, Lemma 4.16], there is a bijection $\varrho_k : V_{i,j} \rightarrow V_{i+k,j-k}$, $-i \leq k \leq j$. Then $\dim(V_{i,j}) = \dim(V_{i+k,j-k})$, since if $v \neq 0$ in $V_{i,j}$ but $\varrho_k(v) = 0$ in $V_{i+k,j-k}$ then $v = \varrho_k^{-1}\varrho_k(v) = \varrho_k^{-1}(0) = 0$ which is a contradiction. The following Lemma shows that an irreducible Hilbert P -module H is determined by its lowest weight modules, and in particular H is determined by its lowest weight module $H_{0,\text{wt}(H)}$, since for all other $i + j = \text{wt}(H)$, $H_{i,j} = \varrho_i(H_{0,\text{wt}(H)})$.

Lemma 4.11 (cf. [14, Lemma 3.7]) *Let P be an A_2 - C^* -planar algebra and let $H^{(1)}, H^{(2)}$ be Hilbert P -modules with $H^{(1)}$ irreducible. Suppose there is a non-zero A_2 - $AP_{i,j}$ homomorphism $\theta : H_{i,j}^{(1)} \rightarrow H_{i,j}^{(2)}$. Then θ extends to an injective homomorphism Θ of P -modules.*

We will now determine which A_2 - $AP_{i,j}$ -modules can be lowest weight modules.

Lemma 4.12 *Let P be an A_2 - C^* -planar algebra and H a Hilbert P -module of lowest weight k and rank (t_1, t_2) . With $i + j = k$, any element $w \in H_{i,j}$ can be written, up to a scalar, as aw for some $a \in A_2$ - $AP_{i,j}$ with $\text{rank}(a) = \text{rank}(w)$.*

Proof: First form $ww^* \in A_2$ - $AP_{i,j}$. Then dividing out by the relations K1-K3 we obtain a linear combination of elements in A_2 - $AP_{i,j}$, and we remove those elements that have $\text{rank} < (t_1, t_2)$. Ignoring the scalar factor we are left with a single element $a \in A_2$ - $AP_{i,j}$ with $\text{rank}(a) = (t_1, t_2)$. If we form aw , then dividing out by K1-K3 we obtain $aw = \mu w + \sum_i \mu_i w_i$, where $\mu, \mu_i \in \mathbb{C}$ and $w_i \in H_{i,j}$ with $\text{rank}(w_i) < (t_1, t_2)$ for each i . Then in H the w_i are all zero, so that $\mu^{-1}aw = w$. \square

Lemma 4.13 (cf. [14, Lemma 3.8]) *Let P be an A_2 - C^* -planar algebra and H a Hilbert P -module. For $3(i+j-1) < 2t'_{\max} + t'_{\min} \leq 3(i+j)$, let $H_{i,j}^{(t_1, t_2)}$ be the A_2 - $AP_{i,j}$ -submodule of $H_{i,j}$ spanned by the i, j -graded pieces of all P -submodules with $\text{rank} < (t_1, t_2)$. Then*

$$(H_{i,j}^{(t_1, t_2)})^\perp = \bigcap_{a \in A_2\text{-}AP_{i,j}^{(t_1, t_2)}} \ker(a).$$

Corollary 4.14 (cf. [14, Cor. 3.10]) *The lowest weight modules of an irreducible P -module of rank (t_1, t_2) are A_2 - $AP_{i,j}/A_2$ - $AP_{i,j}^{(t_1, t_2)}$ -modules, where $2t_{\max} + t_{\min} = 3(i+j)$.*

Then for an A_2 - C^* -planar algebra P , we can determine all Hilbert P -modules by first determining the algebras A_2 - $AP_{0,j}/A_2$ - $AP_{0,j}^{(t_1, t_2)}$ and their irreducible modules, for $2t_{\max} + t_{\min} = 3j$, and then determining which of these modules extend to P -modules.

4.2 Irreducible A_2 - STL -modules

We can easily determine certain irreducible A_2 - STL -modules. We will describe some zero-weight modules. However we have not determined all irreducible A_2 - STL -modules, even for the zero-weight case, since it is not clear that elements of the form $\sigma_{(k_1, k_2)}^{(i)}$ defined below must necessarily contribute a scalar factor, as A_2 - $ATL_{\bar{a}}$ is not one-dimensional (and hence not isomorphic to \mathbb{C}).

Proposition 4.15 (cf. [14, Prop. 5.9]) *The algebra A_2 - $ATL_{\bar{a}}$, for any $a \in \{0, 1, 2\}$, is generated by the 0-tangles $\sigma_{j, j\pm 1}$ illustrated in Figure 5, $j \in \{0, 1, 2\}$.*

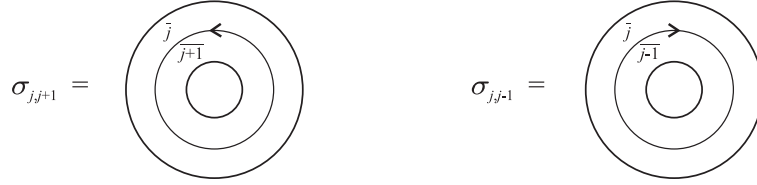


Figure 5: $\sigma_{j, j+1}$ and $\sigma_{j, j-1}$

Let H be an irreducible Hilbert A_2 - STL -module of lowest weight zero. For $i \in \{0, 1, 2\}$ and non-negative integers $k_1 \equiv k_2 \pmod{3}$, let

$$\sigma_{(k_1, k_2)}^{(i)} = (\sigma_{i, i+1} \sigma_{i+1, i+2} \cdots \sigma_{i+k_1-1, i+k_1}) (\sigma_{i+k_1, i+k_1-1} \sigma_{i+k_1-1, i+k_1-2} \cdots \sigma_{i+k_1-k_2+1, i+k_1-k_2}).$$

If the maps $\sigma_{(k_1, k_2)}^{(i)}$ for any k_1, k_2 just give the complex number $\beta^{k_1} \bar{\beta}^{k_2}$ for some fixed $\beta \in \mathbb{C}$, i.e. $\sigma_{(k_1, k_2)}^{(i)} = \beta^{k_1} \bar{\beta}^{k_2} \mathbf{1}_{\bar{i}}$, then the dimensions of $H_{\bar{a}}$ are at most 5, for $a \in \{0, 1, 2\}$. To see this consider an arbitrary element given by a product of elements $\sigma_{j, j\pm 1}$. Whenever the product $\sigma_{l, l+1} \sigma_{l+1, l}$ appears, for some $l \in \{0, 1, 2\}$, we get a factor of $|\beta|^2$. Removing all such products we will be left with an element which contains only non-contractible circles with the same orientation. Any three consecutive such circles contribute a factor of β^3 or $\bar{\beta}^3$. Then up to some scalar, any element will have at most two non-contractible circles, with each circle having the same orientation.

Proposition 4.16 (cf. [14, Theorem 5.12]) *An irreducible Hilbert A_2 - STL -module H of weight zero in which the maps $\sigma_{(k_1, k_2)}^{(i)}$, $i \in \{0, 1, 2\}$, are given by the complex number $\beta^{k_1} \bar{\beta}^{k_2}$ for some fixed $\beta \in \mathbb{C}$ is determined up to isomorphism by the dimensions of $H_{\bar{a}}$, $a \in \{0, 1, 2\}$, and the number β , where we require $|\beta| \leq \alpha$.*

Proof: The uniqueness of the A_2 - STL -module is a consequence of Lemma 4.11 since at least one of $H_{\bar{0}}$, $H_{\bar{1}}$ and $H_{\bar{2}}$ is non-zero. Let E_1, E_2 be the tangles



so that $\alpha^{-1}E_1, \alpha^{-1}E_2$ are projections. Then since $E_1E_2E_1 = |\beta|^2E_1$ we have $\|\alpha^{-1}E_1 \cdot \alpha^{-1}E_2 \cdot \alpha^{-1}E_1\| = |\beta|^2\alpha^{-2}\|\alpha^{-1}E_1\|$ so that $1 \geq |\beta|^2\alpha^{-2}$. Hence $|\beta| \leq \alpha$. \square

For $\beta = \alpha$, $V_{i,j}^\alpha = STL_{i,j}$ (since when $\beta = \alpha$ there is no distinction between contractible and non-contractible circles). For $\alpha > 3$ (which corresponds to $\delta > 2$), the inner product is positive definite by [7, Lemma 3.10] and [27, Theorem 3.6], and $H_{i,j}^\alpha = V_{i,j}^\alpha$ is a Hilbert A_2 - STL -module. For $0 < \alpha \leq 3$, if the inner product is positive semi-definite on $V_{i,j}^\alpha$ we let $H_{i,j}^\alpha$ be the quotient of $V_{i,j}^\alpha$ by the subspace of vectors of length zero; otherwise $H_{i,j}^\alpha$ does not exist.

Now consider the case when $0 < |\beta| < \alpha$. We define for each $i, j \geq 0$ (with 0, 0 replaced by \bar{a} , $a \in \{0, 1, 2\}$, as usual), the set $Th_{i,j}$ to be the set of all $(i, j : \bar{0})$ -tangles with no contractible circles and at most two non-contractible circles. Now for each β we form the graded vector space V^β , where $V_{i,j}^\beta$ has basis $Th_{i,j}$, and we equip it with an A_2 - STL -module structure as follows. Let $T \in A_2\text{-}ATL(i', j' : i, j)$ and $R \in A_2\text{-}ATL_{i,j}$. We form the tangle TR and reduce it using K1-K3, so that $TR = \sum_j \delta^{b_j} \alpha^{c_j} TR_j$, for some basis A_2 -annular $(i', j' : \bar{0})$ -tangles TR_j , where b_j, c_j are non-negative integers. Let \sharp_j^a, \sharp_j^c denote the number of non-contractible circles in the tangle TR_j which have anti-clockwise, clockwise orientation respectively. We define integers d_j, f_j and g_j as follows: $d_j = \min(\sharp_j^a, \sharp_j^c)$, $f_j = \sharp_j^a - \sharp_j^c - \gamma_{f_j}$ if $\sharp_j^a \geq \sharp_j^c$ and $f_j = 0$ otherwise, and $g_j = \sharp_j^c - \sharp_j^a - \gamma_{g_j}$ if $\sharp_j^a \leq \sharp_j^c$ and $g_j = 0$ otherwise, where $\gamma_{f_j}, \gamma_{g_j} \in \{0, 1, 2\}$ such that $f_j, g_j \equiv 0 \pmod{3}$. Then we set $T(R) = \sum_j \delta^{b_j} \alpha^{c_j} \beta^{d_j+f_j} \bar{\beta}^{d_j+g_j} \widehat{TR}_j$, where \widehat{TR}_j is the tangle TR_j with $d_j + f_j$ anti-clockwise non-contractible circles removed, and $d_j + g_j$ clockwise ones removed.

Proposition 4.17 *The above definition make V^β into an A_2 - STL -module of weight zero in which $\sigma_{(k_1, k_2)}^{(a)} = \beta^{k_1} \bar{\beta}^{k_2}$ for $a = 0, 1, 2$.*

As with the A_1 situation, the choice of $(i, j : \bar{0})$ -tangles rather than $(i, j : \bar{1})$ - or $(i, j : \bar{2})$ -tangles to define V^β was arbitrary. For these other two choices, the maps $T \rightarrow \beta^{-1}T\sigma_{01}$, $T \rightarrow \bar{\beta}^{-1}T\sigma_{02}$ respectively would define isomorphisms from those modules to the one defined above.

Definition 4.18 (cf. [14, Def. 5.17]) *Given $S, T \in Th_{i,j}$, we reduce T^*S using K1-K3 so that $T^*S = \sum_j \delta^{b_j} \alpha^{c_j} (T^*S)_j$ for basis $(\bar{0} : \bar{0})$ -tangles $(T^*S)_j$. Define d_j, f_j and g_j for each $(T^*S)_j$ as above. We define an inner-product by $\langle S, T \rangle = \sum_j \delta^{b_j} \alpha^{c_j} \beta^{d_j+f_j} \bar{\beta}^{d_j+g_j}$.*

Invariance of this inner-product follows from the fact that $T^*S = \langle S, T \rangle T_0$ where T_0 is the annular $(\bar{0} : \bar{0})$ -tangle with no strings at all. When the above inner-product is positive semi-definite, we define the Hilbert A_2 - STL -module H^β of weight zero to be the quotient of V^β by the subspace of vectors of length zero. Otherwise H^β does not exist.

Proposition 4.19 *For the above Hilbert A_2 - STL -module H^β of weight zero, the dimension of H_a^β is either 0 or 1 for any $\beta \in \mathbb{C} \setminus \{0\}$.*

Proof: For $a = 0$ the result is trivial since V_0^β is the linear span of the empty tangle T_0 given in Definition 4.18. For $a = 1$, $V_1^\beta = \text{span}(\sigma_{10}, \sigma_{12}\sigma_{20})$. Let $w = |\beta|^2\sigma_{12}\sigma_{20} - \beta^3\sigma_{10}$. Then

$$\begin{aligned} \langle w, w \rangle &= |\beta|^4 \langle \sigma_{12}\sigma_{20}, \sigma_{12}\sigma_{20} \rangle - |\beta|^2 \bar{\beta}^3 \langle \sigma_{12}\sigma_{20}, \sigma_{10} \rangle - \beta^3 |\beta|^2 \langle \sigma_{10}, \sigma_{12}\sigma_{20} \rangle + |\beta|^6 \langle \sigma_{10}, \sigma_{10} \rangle \\ &= |\beta|^4 (|\beta|^4) - |\beta|^2 \bar{\beta}^3 \beta^3 - \beta^3 |\beta|^2 \bar{\beta}^3 + |\beta|^6 |\beta|^2 = 0. \end{aligned}$$

Then $\sigma_{10} = |\beta|^2 \beta^{-3} \sigma_{12} \sigma_{20} = \bar{\beta} \beta^{-2} \sigma_{12} \sigma_{20}$ in H_1^β . Similarly when $a = 2$, $\sigma_{21} \sigma_{10} = \bar{\beta}^2 \beta^{-1} \sigma_{20}$. \square

So we may define H^β so that it does not contain any clockwise non-contractible circles, where we replace every σ_{10} by $\bar{\beta} \beta^{-2} \sigma_{12} \sigma_{20}$ and every $\sigma_{21} \sigma_{10}$ by $\bar{\beta}^2 \beta^{-1} \sigma_{20}$.

Proposition 4.20 (cf. [14, Cor. 5.8]) *The Hilbert A_2 -STL-module H^β , $|\beta| < \alpha$, is irreducible.*

Proof: Since H_a^β is at most one-dimensional it must be irreducible, for each $a \in \{0, 1, 2\}$. The maps $\sigma_{j,j+1}$ moves a non-zero element in H_j^β to an element in H_{j+1}^β , and hence the lowest weight module $H_0^\beta = H_0^\beta \oplus H_1^\beta \oplus H_2^\beta$ is irreducible as an A_2 -ATL₀-module. Since $H^\beta = A_2\text{-ATL}(H_0^\beta)$, the result follows from Lemma 4.9. \square

Now we consider the case when $\beta = 0$. For each $i, j \geq 0$ (with $0, 0$ replaced by \bar{a} , $a \in \{0, 1, 2\}$, as usual), the set $Th_{i,j}^{\bar{a}}$ is defined to be the set of all $(i, j : \bar{a})$ -tangles with no contractible or non-contractible circles at all. The cardinality of $Th_{i,j}^{\bar{a}}$ is $\delta_{a,b}$. We form the graded vector space $V^{0,\bar{a}}$, where $V_{i,j}^{0,\bar{a}}$ has basis $Th_{i,j}^{\bar{a}}$. We equip it with an A_2 -STL-module structure of lowest weight zero as follows. Let $T \in A_2\text{-ATL}(i', j' : i, j)$ and $R \in Th_{i,j}^{\bar{a}}$. We form TR and reduce it using K1-K3, so that $TR = \sum_j \delta^{b_j} \alpha^{c_j}$ as in the case $0 < |\beta| < \alpha$. We define $T(R)_j$ to be zero if there are any non-contractible circles in TR_j , and TR_j otherwise. Then $T(R) = \sum_j \delta^{b_j} \alpha^{c_j} T(R)_j$.

Proposition 4.21 *The above definition make $V^{0,\bar{a}}$ into an A_2 -STL-module of weight zero in which $\sigma_{j,j\pm 1} = 0$ for $j = 0, 1, 2 \bmod 3$.*

Definition 4.22 (cf. [14, Def. 5.22]) *Given $S, T \in Th_{i,j}^{\bar{a}}$, we reduce T^*S using K1-K3 so that $T^*S = \sum_j \delta^{b_j} \alpha^{c_j} (T^*S)_j$ for basis $(\bar{a} : \bar{a})$ -tangles $(T^*S)_j$. We define $\langle S, T \rangle_j$ to be 0 if there are any non-contractible circles in $(T^*S)_j$, and 1 otherwise. Then we define an inner-product by $\langle S, T \rangle = \sum_j \delta^{b_j} \alpha^{c_j} \langle S, T \rangle_j$.*

This inner-product is invariant as in the case $0 < |\beta| < \alpha$. Again, if the inner product is positive semi-definite we define $H^{0,\bar{a}}$ to be the quotient of $V^{0,\bar{a}}$ by the subspace of vectors with length zero; otherwise $H^{0,\bar{a}}$ does not exist.

Proposition 4.23 *The Hilbert A_2 -STL-module $H^{0,\bar{a}}$, $a \in \{0, 1, 2\}$, is irreducible.*

Proof is as for H^β .

5 The A_2 -planar algebra of an \mathcal{ADE} graph

Let \mathcal{G} be any finite $SU(3)$ \mathcal{ADE} graph (not necessarily one for which there exists a flat connection) with vertex set $\mathfrak{V}^\mathcal{G} = \mathfrak{V}_0^\mathcal{G} \cup \mathfrak{V}_1^\mathcal{G} \cup \mathfrak{V}_2^\mathcal{G}$, where $\mathfrak{V}_a^\mathcal{G}$ is the set of a -coloured vertices of \mathcal{G} , $a = 0, 1, 2$. Let $n_a = |\mathfrak{V}_a^\mathcal{G}|$ denote the number of a -coloured vertices and $n = |\mathfrak{V}^\mathcal{G}| = n_0 + n_1 + n_2$ the total number of vertices of \mathcal{G} . Note that $n_1 = n_2$ due to the conjugation property of the $SU(3)$ \mathcal{ADE} graphs. For the non-three-colourable graphs $n_1 = n_2 = 0$ and $n = n_0$. Let $\alpha = [3]_q$, $q = e^{i\pi/n}$, be the Perron-Frobenius eigenvalue of \mathcal{G} and let $\phi = (\phi_v)$ be the corresponding eigenvector.

We define a double sequence

$$\begin{array}{cccc}
C_{0,0} & \subset & C_{0,1} & \subset & C_{0,2} & \subset & \cdots \\
\cap & & \cap & & \cap & & \\
C_{1,0} & \subset & C_{1,1} & \subset & C_{1,2} & \subset & \cdots \\
\cap & & \cap & & \cap & & \\
C_{2,0} & \subset & C_{2,1} & \subset & C_{2,2} & \subset & \cdots \\
\cap & & \cap & & \cap & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

where $C_{0,0} = \mathbb{C}^{n_0}$. The Bratteli diagrams for horizontal inclusions $C_{i,j} \subset C_{i,j+1}$ are given by \mathcal{G} . If \mathcal{G} is three-colourable, the vertical inclusions $C_{i,j} \subset C_{i+1,j}$ are given by its $\overline{j, j+1}$ -part $\mathcal{G}_{\overline{j, j+1}}$, where $\overline{p} = \tau(p)$ is the colour of p for $p = j, j+1$, whilst if \mathcal{G} is not three-colourable we use the graph \mathcal{G} for all the vertical inclusions $C_{i,j} \subset C_{i+1,j}$. Then for the inclusions

$$\begin{array}{ccc}
C_{i,j} & \subset & C_{i,j+1} \\
\cap & & \cap \\
C_{i+1,j} & \subset & C_{i+1,j+1}
\end{array} \tag{9}$$

with i even, we define a connection by

$$X_{\sigma_3, \sigma_4}^{\sigma_1, \sigma_2} = \sigma_3 \downarrow \begin{array}{c} \xrightarrow{\sigma_1} \\ \downarrow \sigma_2 \\ \xrightarrow{\sigma_4} \end{array} = q^{\frac{2}{3}} \delta_{\sigma_1, \sigma_3} \delta_{\sigma_2, \sigma_4} - q^{-\frac{1}{3}} \mathcal{U}_{\sigma_3, \sigma_4}^{\sigma_1, \sigma_2}, \tag{10}$$

We denote by $\tilde{\mathcal{G}}$ the reverse graph of \mathcal{G} , which is the graph obtained by reversing the direction of every edge of \mathcal{G} . For the inclusions (9) with i odd, let σ_1, σ_4 be edges on \mathcal{G} and let $\tilde{\sigma}_2, \tilde{\sigma}_3$ be edges on the reverse graph $\tilde{\mathcal{G}}$ (so that σ_2, σ_3 are edges on \mathcal{G}). We define the connection by

$$X_{\tilde{\sigma}_3, \sigma_4}^{\sigma_1, \tilde{\sigma}_2} = \tilde{\sigma}_3 \downarrow \begin{array}{c} \xrightarrow{\sigma_1} \\ \downarrow \tilde{\sigma}_2 \\ \xrightarrow{\sigma_4} \end{array} = \sqrt{\frac{\phi_{s(\sigma_3)} \phi_{r(\sigma_2)}}{\phi_{r(\sigma_3)} \phi_{s(\sigma_2)}}} \overline{\sigma_3 \downarrow \begin{array}{c} \xrightarrow{\sigma_4} \\ \downarrow \sigma_2 \\ \xrightarrow{\sigma_1} \end{array}}. \tag{11}$$

It was shown in [5] that these connections satisfy the unitarity axiom

$$\sum_{\sigma_3, \sigma_4} X_{\sigma_3, \sigma_4}^{\sigma_1, \sigma_2} \overline{X_{\sigma_3, \sigma_4}^{\sigma'_1, \sigma'_2}} = \delta_{\sigma_1, \sigma'_1} \delta_{\sigma_2, \sigma'_2}. \tag{12}$$

We make the double sequence $(C_{i,j})$ into an A_2 - C^* -planar algebra as follows. Let $P_a^{\mathcal{G}} = \mathfrak{V}_a^{\mathcal{G}}$ for $i = j = 0$, and $P_{i,j}^{\mathcal{G}} = C_{i,j}$ for all other $i, j \geq 0$. We define a presenting map $Z : \mathcal{P}(P^{\mathcal{G}}) \rightarrow P_{i,j}^{\mathcal{G}}$ in the same way as we did for the double sequence $(B_{i,j})$ of finite dimensional algebras with a flat connection in [7]. First, convert all the discs D_k to rectangles, with the first $i_k + j_k$ vertices along one edge, and the next $i_k + j_k$ vertices along the opposite edge, and rotate each rectangle so that those edges are horizontal with the first vertex on the top edge. Next, isotope the strings of T so that each horizontal strip only contains one of the following elements: a rectangle with label x_k , a cup, a cap, a Y-fork, or an inverted Y-fork. Let C be the set of all strips containing one of these

elements except for a labelled rectangle. We will use the following notation for elements of C , as shown in Figures 6, 7 and 8: A strip containing a cup, cap will be $\cup^{(i)}$, $\cap^{(i)}$ respectively, where there are $i - 1$ vertical strings to the left of the cup or cap. Strips containing an incoming Y-fork, inverted Y-fork will be $\gamma^{(i)}$, $\lambda^{(i)}$ respectively, where there are $i - 1$ vertical strings to the left of the (inverted) Y-fork. A bar will denote that it is an outgoing (inverted) Y-fork.



Figure 6: Cup $\cup^{(i)}$ and cap $\cap^{(i)}$



Figure 7: Y-forks $\gamma^{(i)}$ and $\bar{\gamma}^{(i)}$



Figure 8: Inverted Y-forks $\lambda^{(i)}$ and $\bar{\lambda}^{(i)}$

For an element $c \in C$ with n_1, n_2 strings having endpoints (we will call these endpoints vertices) along the top, bottom edge respectively of the strip, let the orientations of these vertices along the top, bottom edge respectively of the strip be given by the sequences $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$ respectively, where for $i = 1, 2$, $\mathbf{v}^{(i)} = (v_0^{(i)}, v_1^{(i)}, v_2^{(i)}, \dots, v_{l_i}^{(i)})$, where $v_0^{(i)} \in \mathbb{N} \cup \{0\}$ and $v_k^{(i)} \in \mathbb{N}$ for $k \geq 1$, with $\sum_{k=1}^{l_i} v_k^{(i)} = n_i$. The numbers $v_k^{(i)}$ denote the number of consecutive vertices with downwards, upwards orientation for k even, odd respectively. Note that if the first vertex along the top, bottom of the strip has upwards orientation, then $v_0^{(i)} = 0$ for $i = 1, 2$ respectively. The leftmost region of the strip c corresponds to the vertex $*$ of \mathcal{G} , and each vertex along the top (or bottom) with downwards, upwards orientation respectively, corresponds to an edge on \mathcal{G} , $\tilde{\mathcal{G}}$ respectively ($\tilde{\mathcal{G}}$ is the graph \mathcal{G} with all orientations reversed). Then the top, bottom edge of the strip corresponds is labelled by all paths on \mathcal{G} and $\tilde{\mathcal{G}}$ which start at $*$ and have the form given by $\mathbf{v}^{(i)}$. These paths are uniquely described by the sequence of edges they pass along. Let H_1, H_2 be the Hilbert spaces corresponding to all paths of the form $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}$ respectively. Then $Z(c)$ defines an operator $M_c \in \text{End}(H_1, H_2)$ as follows.

For a cup $\cup^{(i)}$, and paths $\alpha = \alpha_1 \cdot \alpha_2 \cdots \alpha_j$, $\beta = \beta_1 \cdots \beta_{j+2}$,

$$(M_{\cup^{(i)}})_{\alpha,\beta} = \delta_{\alpha_1,\beta_1} \delta_{\alpha_2,\beta_2} \cdots \delta_{\alpha_{i-1},\beta_{i-1}} \delta_{\alpha_i,\beta_{i+2}} \delta_{\alpha_{i+1},\beta_{i+3}} \cdots \delta_{\alpha_m,\beta_{m+2}} \delta_{\tilde{\beta}_i,\beta_{i+1}} \frac{\sqrt{\phi_{r(\beta_i)}}}{\sqrt{\phi_{s(\beta_i)}}}. \quad (13)$$

For a cap $\cap^{(i)}$,

$$M_{\cap^{(i)}} = M_{\cup^{(i)}}^*. \quad (14)$$

For an incoming (inverted) Y-fork $\gamma^{(i)}$ or $\lambda^{(i)}$,

$$(M_{\gamma^{(i)}})_{\alpha,\beta} = \delta_{\alpha_1,\beta_1} \cdots \delta_{\alpha_{i-1},\beta_{i-1}} \delta_{\alpha_{i+1},\beta_{i+2}} \cdots \delta_{\alpha_m,\beta_{m+1}} \frac{1}{\sqrt{\phi_{s(\alpha_i)} \phi_{r(\alpha_i)}}} W(\Delta_{(\tilde{\alpha}_i \cdot \beta_i \cdot \beta_{i+1})}), \quad (15)$$

$$(M_{\lambda^{(i)}})_{\alpha,\beta} = \delta_{\alpha_1,\beta_1} \cdots \delta_{\alpha_{i-1},\beta_{i-1}} \delta_{\alpha_{i+2},\beta_{i+1}} \cdots \delta_{\alpha_{m+1},\beta_m} \frac{1}{\sqrt{\phi_{s(\beta_i)} \phi_{r(\beta_i)}}} \overline{W(\Delta_{(\beta_i \cdot \tilde{\alpha}_{i+1} \cdot \tilde{\alpha}_i)})}, \quad (16)$$

where W is a cell system on \mathcal{G} (see [23, 5]).

For an outgoing (inverted) Y-fork $\overline{\gamma}^{(i)}$ or $\overline{\lambda}^{(i)}$,

$$M_{\overline{\gamma}^{(i)}} = M_{\lambda^{(i)}}^*, \quad (17)$$

$$M_{\overline{\lambda}^{(i)}} = M_{\gamma^{(i)}}^*. \quad (18)$$

For a strip containing a rectangle with label $x_k = \sum_{\gamma,\gamma'} \lambda_{\gamma,\gamma'}(\gamma, \gamma')$ where $\lambda_{\gamma,\gamma'} \in \mathbb{C}$ and $(\gamma, \gamma') \in P_{i_k,j_k}$ are matrix units indexed by paths γ, γ' , we define the operator $M_{b_k} = Z(b_k)$ as follows. Let p_k, p'_k be the number of vertical strings to the left, right of the rectangle in strip b_k respectively, with orientations given by the sequences $\mathbf{v}^{(p_k)} = (v_0^{(p_k)}, v_1^{(p_k)}, \dots, v_{p_k}^{(p_k)})$, $\mathbf{v}^{(p'_k)} = (v_0^{(p'_k)}, v_1^{(p'_k)}, \dots, v_{p'_k}^{(p'_k)})$ respectively. Then $\sum_{\gamma,\gamma',\mu_i} \lambda_{\gamma,\gamma'}(\mu_1 \cdot \gamma \cdot \mu_2, \mu_1 \cdot \gamma' \cdot \mu_2)$ defines a matrix M_{b_k} , where the summation is over all paths μ_1 on \mathcal{G} , $\tilde{\mathcal{G}}$ of length p_k of the form $\mathbf{v}^{(p_k)}$, and paths μ_2 on \mathcal{G} , $\tilde{\mathcal{G}}$ of length p'_k of the form $\mathbf{v}^{(p'_k)}$. For a tangle $T \in \mathcal{P}_{i,j}$ with l horizontal strips s_l , where s_1 is the lowest strip, s_2 the strip immediately above it, and so on, we define $Z(T) = Z(s_1)Z(s_2) \cdots Z(s_l)$, which will be an element of $P_{i,j}^{\mathcal{G}}$.

Theorem 5.1 *The above definition of $Z(T)$ for any A_2 -planar tangle T makes the double sequence $(C_{i,j})$ into an A_2 - C^* -planar algebra $P^{\mathcal{G}}$, with $\dim(P_a^{\mathcal{G}}) = n_a$, $a = 0, 1, 2$, and parameter [3].*

Proof: This follows as in the proof of Theorem 5.4 in [7], where the only small difference occurs for isotopies of the tangle which involve rectangles. However the invariance is simpler here as the connection is not used. \square

The partition functions $Z : \mathcal{P}_{\bar{a}} \rightarrow \mathbb{C}$ are defined as the linear extensions of the function which takes the basis path v to ϕ_v^2 . There is an extra multiplicative factor of ϕ_v^2 for the external region. This is required for spherical isotopy.

Proposition 5.2 (cf. [13, Prop. 3.4]) *The partition function of a closed labelled tangle T depends only on T up to isotopies of the 2-sphere.*

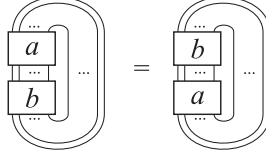


Figure 9: $\text{Tr}(ab) = \text{Tr}(ba)$

We normalize (ϕ_v) so that the partition function of an empty closed tangle is equal to one. We will say that the $\text{SU}(3)$ -planar algebra of a graph \mathcal{G} is **normalized** if

$$\sum_{v \in \mathfrak{V}_0^{\mathcal{G}}} \phi_v^2 = 1.$$

Theorem 5.3 (cf. [13, Theorem 3.6]) *Let $P^{\mathcal{G}}$ be the normalized A_2 -planar algebra of an ADE graph \mathcal{G} , with (normalized) Perron-Frobenius eigenvector (ϕ_v) . Then for $x \in P_{i,j}^{\mathcal{G}}$,*

$$\text{tr}(x) = [3]^{-i-j} Z(\hat{x})$$

*defines a normalized trace on the union of the P 's, where \hat{x} is any 0-tangle obtained from x by connecting the first $i+j$ boundary points to the last $i+j$. The scalar product $\langle x, y \rangle = \text{tr}(x^*y)$ is positive definite.*

Proof: The normalization makes the definition of the trace consistent with the inclusions. The property $\text{tr}(ab) = \text{tr}(ba)$ is a consequence of planar isotopy when all the strings added to x to get \hat{x} go round x in the same direction, as in Figure 9. Spherical isotopy reduces the general case to the one above. Positive definiteness follows from the fact that the matrix units $e = (\gamma, \gamma') \in P_{i,j}^{\mathcal{G}}$ are mutually orthogonal elements of positive length: $\langle e, e \rangle = [3]^{-i-j} \phi_{v_1} \phi_{v_2} > 0$, where $e \in P_{i,j}^{\mathcal{G}}$ is a pair of paths of length $i+j$ starting at vertex v_1 and ending at vertex v_2 , and $\phi_v > 0$ for all v since ϕ is a Perron-Frobenius eigenvector. \square

5.1 $P^{\mathcal{G}}$ as an A_2 - STL -module

Let $\Delta_{\mathcal{G}}$ denote the adjacency matrix of the graph \mathcal{G} . If \mathcal{G} is three-colourable then $\Delta_{\mathcal{G}}$ may be written in the form

$$\Delta_{\mathcal{G}} = \begin{pmatrix} 0 & \Delta_{01} & 0 \\ 0 & 0 & \Delta_{12} \\ \Delta_{20} & 0 & 0 \end{pmatrix},$$

where Δ_{01} , Δ_{12} and Δ_{20} are matrices which give the number of edges between each 0,1,2-coloured vertex respectively of \mathcal{G} to each 1,2,0-coloured vertex respectively. By a suitable ordering of the vertices the matrix Δ_{12} may be chosen to be symmetric. These matrices satisfy the conditions $\Delta_{01}^T \Delta_{01} = \Delta_{20} \Delta_{20}^T = \Delta_{12}^2$, $\Delta_{01} \Delta_{01}^T = \Delta_{20}^T \Delta_{20}$, which follow from the fact that $\Delta_{\mathcal{G}}$ is normal [6]. For non-three-colourable \mathcal{G} , we define $\Delta_{01} = \Delta_{12} = \Delta_{20} = \Delta_{\mathcal{G}}$. Let $\Lambda_{i,j}^1$, $\Lambda_{i,j}^2$ be the product of j , i matrices respectively, given by

$$\Lambda_{i,j}^1 = \Delta_{01} \Delta_{12} \Delta_{20} \Delta_{01} \cdots \Delta_{\overline{j-1}, \overline{j}}, \quad \Lambda_{i,j}^2 = \Delta_{\overline{j}, \overline{j+1}} \Delta_{\overline{j}, \overline{j+1}}^T \Delta_{\overline{j}, \overline{j+1}} \Delta_{\overline{j}, \overline{j+1}}^T \cdots \Delta',$$

where Δ' is $\Delta_{\overline{j}, \overline{j+1}}$ if i is odd, $\Delta_{\overline{j}, \overline{j+1}}^T$ if i is even, and \overline{p} is the colour of p . Note that $\Lambda_{i,j}^1$ is a normal operator since $\Lambda_{i,j}^1(\Lambda_{i,j}^1)^* = \Lambda_{i,j}^1(\Lambda_{i,j}^1)^T = (\Delta_{01}\Delta_{01}^T)^j$ and $(\Lambda_{i,j}^1)^*\Lambda_{i,j}^1 = (\Lambda_{i,j}^1)^T\Lambda_{i,j}^1 = (\Delta_{01}\Delta_{01}^T)^j$. Similarly $\Lambda_{i,j}^2$ is a normal operator.

Let β_l^3 , $l \in \mathfrak{V}_0^{\mathcal{G}}$, be the eigenvalues of $\Lambda_{i,3}^1$, and $v^{(l)}$ their corresponding eigenvectors. Then $(\Lambda_{i,3}^1)^T v^{(l)} = \overline{\beta}_l^3 v^{(l)}$ and $(\Delta_{01}\Delta_{01}^T)^3 v^{(l)} = \Lambda_{i,j}^1(\Lambda_{i,j}^1)^T v^{(l)} = |\beta_j|^6 v^{(l)}$. Then if λ_l are the eigenvalues of $\Delta_{01}\Delta_{01}^T$ with corresponding eigenvectors $v^{(l)'}$, $l \in \mathfrak{V}_0^{\mathcal{G}}$, we have $(\Delta_{01}\Delta_{01}^T)^3 v^{(l)'} = \lambda_l^3 v^{(l)'}$ so that $v^{(l)'} = v^{(l)}$ and $\lambda_l = |\beta_l|^2$.

Let $n' = \min\{n_0, n_1\}$. The dimension of $P_{i,j}^{\mathcal{G}}$ is given by the trace of $\Lambda\Lambda^T$ where $\Lambda = (\Lambda_{i,j}^1)^i(\Lambda_{i,j}^2)^j$, which counts the number of pairs of paths on $\mathcal{G}, \tilde{\mathcal{G}}$. Since $\Lambda\Lambda^T = (\Delta\Delta^T)^{i+j}$, the trace of $\Lambda\Lambda^T$ is given by the sum $\sum_l \nu_{i,j}^{(l)}$ of its eigenvalues, $l = 1, 2, \dots, n'$. The eigenvalues $\nu_{i,j}^{(l)}$ are given by $|\beta_l|^{2(i+j)}$, where β_l^3 are the eigenvalues of $\Lambda_{i,3}^1$. The Hilbert series for $P^{\mathcal{G}}$ is then given by

$$\Phi_{P^{\mathcal{G}}}(z_1, z_2) = \frac{1}{3}(n_0 + 2n_1 - 3n') + \sum_{l=1}^{n'} \frac{1}{(1 - |\beta_l|^2 z_1)(1 - |\beta_l|^2 z_2)}.$$

Proposition 5.4 (cf. [26, Prop. 13]) *Let \mathcal{G} be one of the finite $SU(3)$ ADE graphs, let ζ_l , $l = 1, 2, \dots, n'$, be the non-zero eigenvalues of $\Lambda_{0,3}^1$, counting multiplicity, and let β_l be any cubic root of ζ_l , $l = 1, 2, \dots, n'$. For all the three-colourable graphs except $\mathcal{E}_5^{(12)}$, we have $n_0 \geq n_1$, all the irreducible weight-zero A_2 -ATL-submodules of $P^{\mathcal{G}}$ are H^{β_l} , $l = 1, 2, \dots, n_1$, and $(n_0 - n_1)$ copies of H^0 , and these can be assumed to be mutually orthogonal. For $\mathcal{E}_5^{(12)}$ we have $n_1 > n_0$, and all the irreducible weight-zero A_2 -ATL-submodules of $P^{\mathcal{E}_5^{(12)}}$ are H^{β_l} , $l = 1, 2, \dots, n_0$, and $2(n_1 - n_0)$ copies of H^0 , which can again be assumed to be mutually orthogonal. If \mathcal{G} is not three-colourable, all the irreducible weight-zero A_2 -ATL-submodules of $P^{\mathcal{G}}$ are H^{β_l} , $l = 1, 2, \dots, n_0$, where n_0 is the total number of vertices of \mathcal{G} .*

Proof: Consider the case where $n_0 > n_1$ (the case for $\mathcal{E}_5^{(12)}$ where $n_1 > n_0$ is similar). Each β_l -eigenvector $v^{(l)} = (v_w^{(l)})$, $w \in \mathfrak{V}_0^{\mathcal{G}}$ of $\Delta_{01}\Delta_{01}^T$ spans a one-dimensional subspace of $P_0^{\mathcal{G}}$ that is invariant under A_2 -ATL $_{\overline{0}}$. To see this, first consider the element $\sigma_{01}\sigma_{12}\sigma_{20}$:

$$\sigma_{01}\sigma_{12}\sigma_{20}v^{(l)} = \sigma_{01}\sigma_{12}\sigma_{20} \sum_{w \in \mathfrak{V}_0^{\mathcal{G}}} v_w^{(l)} = \sum_{w', w} (\Delta_{01}\Delta_{12}\Delta_{20})_{w', w} v_w^{(l)},$$

which, by the β_l eigenequation gives

$$\sigma_{01}\sigma_{12}\sigma_{20}v^{(l)} = \sum_{w'} \beta_l^3 v_{w'}^{(l)} = \beta_l^3 v^{(l)}. \quad (19)$$

Similarly for $\sigma_{20}^*\sigma_{12}^*\sigma_{01}^*$. Next consider the general element σ given by the composition of $2k$ elements $\sigma = \sigma_{01}\sigma_{12}\sigma_{20}\sigma_{01} \cdots \sigma_{\overline{k-1}, \overline{k}}^* \sigma_{\overline{k-1}, \overline{k}}^* \cdots \sigma_{12}^* \sigma_{01}^*$:

$$\begin{aligned} \sigma v^{(l)} &= \sum_{w', w} (\Delta_{01}\Delta_{12} \cdots \Delta_{\overline{k-1}, \overline{k}} \Delta_{\overline{k-1}, \overline{k}}^T \cdots \Delta_{01}^T)_{w', w} v_w^{(l)} \\ &= \sum_{w', w} ((\Delta_{01}\Delta_{01}^T)^k)_{w', w} v_w^{(l)} = \sum_{w'} |\beta_l|^{2k} v_{w'}^{(l)} = |\beta_l|^{2k} v^{(l)}. \end{aligned} \quad (20)$$

Any element of $A_2\text{-}ATL_{\bar{0}}$ is a linear combination of products of elements $\sigma_{j,j\pm 1}$ such that the regions which meet the outer and inner boundaries have colour 0. Let σ be such an element. Then the action of σ on the β_l -eigenvector $v^{(l)}$ is given by $\sigma v^{(l)} = \sum_{w',w} M(w',w)v_w^{(l)}$, where M is the product of matrices Δ, Δ^T given by replacing every $\sigma_{j,j+1}, \sigma_{j',j'-1}$ in σ by Δ, Δ^T respectively. Then by (19) and (20), this gives some scalar multiple of $v^{(l)}$. Then each β_l -eigenvector $v^{(l)}$ generates the submodule H^{β_l} by Proposition 4.16. The inner product on H^{β_l} coincides with the inner product on $P^{\mathcal{G}}$. To see this we only need to check its restriction to the zero-weight part because of (8). For any element $A \in A_2\text{-}ATL_{\bar{0}}$, $\langle Av, v \rangle_{H^{\beta_l}} = c \langle v, v \rangle_{H^{\beta_l}}$ whilst $\langle Av^{(l)}, v^{(l)} \rangle_{P^{\mathcal{G}}} = d \langle v^{(l)}, v^{(l)} \rangle_{P^{\mathcal{G}}}$. The element A is necessarily a combination of non-contractible circles, which gives the same contribution in $P^{\mathcal{G}}$ as in H^{β_l} by (19), (20). So $c = d$. This shows that the inner product on the H^{β_l} is positive definite, since the inner product on $P^{\mathcal{G}}$ is.

Similarly, a 0-eigenvector generates the submodule H^0 , where for $n_0 > n_1$, $\dim(H_{\bar{0}}^{0,\bar{0}}) = 1$ and $\dim(H_{\bar{1}}^{0,\bar{1}}) = \dim(H_{\bar{2}}^{0,\bar{2}}) = 0$, whilst for $\mathcal{E}_5^{(12)}$ we have $\dim(H_{\bar{1}}^{0,\bar{1}}) = \dim(H_{\bar{2}}^{0,\bar{2}}) = 1$ and $\dim(H_{\bar{0}}^{0,\bar{0}}) = 0$. As in the $SU(2)$ case, in order to make the resulting submodules orthogonal we take an orthogonal set of eigenvectors. \square

For an \mathcal{ADE} graph \mathcal{G} with Coxeter number n , let $\beta_{(l_1,l_2)}$ be the eigenvalue of \mathcal{G} given by

$$\beta_{(l_1,l_2)} = \exp\left(\frac{2i\pi}{3n}(l_1 + 2l_2 + 3)\right) + \exp\left(-\frac{2i\pi}{3n}(2l_1 + l_2 + 3)\right) + \exp\left(\frac{2i\pi}{3n}(l_1 - l_2)\right) \quad (21)$$

for exponent (l_1, l_2) . Then for the graphs $\mathcal{A}^{(n)}$, we have for $n \not\equiv 0 \pmod{3}$,

$$P^{\mathcal{A}^{(n)}} \supset \bigoplus_{(l_1,l_2)} H^{\beta_{(l_1,l_2)}}, \quad (22)$$

whilst for $n = 3k$, $k \geq 2$,

$$P^{\mathcal{A}^{(3k)}} \supset \bigoplus_{(l_1,l_2)} H^{\beta_{(l_1,l_2)}} \oplus H^{0,\bar{0}}, \quad (23)$$

where in both cases the summation is over all $(l_1, l_2) \in \{(m_1, m_2) \mid 3m_2 \leq n - 3, 3m_1 + 3m_2 \leq 2n - 6\}$, i.e. each $\beta_{(l_1,l_2)}$ is a cubic root of an eigenvalue of $\Lambda_{0,3}^1$. We believe that we in fact have equality here, so that $P^{\mathcal{A}^{(n)}} = \bigoplus_{(l_1,l_2)} H^{\beta_{(l_1,l_2)}}$. In the $SU(2)$ case this was achieved by a dimension count of the left and right hand sides [26, Theorem 15]. However, we have not yet been able to determine a similar result for the $SU(3)$ \mathcal{A} graphs.

For the other \mathcal{ADE} graphs, Proposition 5.4 gives the following results for the zero-weight part of $P^{\mathcal{G}}$. For the \mathcal{D} graphs, we have

$$P^{\mathcal{D}^{(3k)}} \supset \bigoplus_{(l_1,l_2)} H^{\beta_{(l_1,l_2)}} \oplus 3H^{0,\bar{0}}, \quad (24)$$

for $k \geq 2$, where the summation is over all $(l_1, l_2) \in \{(m_1, m_2) \mid m_2 \leq k - 1, m_1 + m_2 \leq 2k - 2, m_1 - m_2 \equiv 0 \pmod{3}\}$, whilst for $n \not\equiv 0 \pmod{3}$,

$$P^{\mathcal{D}^{(n)}} \supset \bigoplus_{(l_1,l_2)} H^{\beta_{(l_1,l_2)}}, \quad (25)$$

where the summation is over all $(l_1, l_2) \in \{(m_1, m_2) | 3m_2 \leq n - 3, 3m_1 + 3m_2 \leq 2n - 6\}$. The path algebras for $\mathcal{A}^{(n)*}$ and $\mathcal{D}^{(n)*}$ are identified under the map which send the vertices i_l, j_l and k_l of $\mathcal{D}^{(n)*}$ with the vertex l of $\mathcal{A}^{(n)*}$, $l = 1, 2, \dots, \lfloor l/2 \rfloor$. We have

$$P^{\mathcal{A}^{(n)*}} = P^{\mathcal{D}^{(n)*}} \supset \bigoplus_{(l_1, l_2)} H^{\beta_{(l_1, l_2)}}, \quad (26)$$

where the summation is over all $(l_1, l_2) \in \{(m, m) | m = 0, 1, \dots, \lfloor (n - 3)/2 \rfloor\}$. Similarly, the path algebras for $\mathcal{E}^{(8)}$ and $\mathcal{E}^{(8)*}$ are identified, and

$$P^{\mathcal{E}^{(8)}} = P^{\mathcal{E}^{(8)*}} \supset H^{\beta_{(0,0)}} \oplus H^{\beta_{(3,0)}} \oplus H^{\beta_{(0,3)}} \oplus H^{\beta_{(2,2)}}. \quad (27)$$

For the graphs $\mathcal{E}_i^{(12)}$, $i = 1, 2, 3$, we have

$$P^{\mathcal{E}_i^{(12)}} \supset H^{\beta_{(0,0)}} \oplus 2H^{\beta_{(2,2)}} \oplus H^{\beta_{(4,4)}}. \quad (28)$$

For the remaining exceptional graphs we have

$$P^{\mathcal{E}_4^{(12)}} \supset H^{\beta_{(0,0)}} \oplus H^{\beta_{(2,2)}} \oplus H^{\beta_{(4,4)}} \oplus 2H^{0, \bar{0}}, \quad (29)$$

$$P^{\mathcal{E}_5^{(12)}} \supset H^{\beta_{(0,0)}} \oplus H^{\beta_{(3,0)}} \oplus H^{\beta_{(0,3)}} \oplus H^{\beta_{(2,2)}} \oplus H^{\beta_{(4,4)}} \oplus H^{0, \bar{1}} \oplus H^{0, \bar{2}}, \quad (30)$$

$$P^{\mathcal{E}^{(24)}} \supset H^{\beta_{(0,0)}} \oplus H^{\beta_{(6,0)}} \oplus H^{\beta_{(0,6)}} \oplus H^{\beta_{(4,4)}} \oplus H^{\beta_{(7,4)}} \oplus H^{\beta_{(4,7)}} \oplus H^{\beta_{(6,6)}} \oplus H^{\beta_{(10,10)}}. \quad (31)$$

The A_2 -planar algebra $P \cong STL$ of [7] (for the graphs $\mathcal{A}^{(n)}$) clearly have decomposition $P = H^\alpha$ as an A_2 - ATL -module, since STL is equal to the A_2 - ATL -module H^α (see Section 4.2). Since every A_2 -planar algebra contains STL , the A_2 -planar algebra for all the \mathcal{ADE} graphs with a flat connection will contain the zero-weight module H^α .

5.2 Irreducible modules with non-zero weight

We will now present some conjectured irreducible A_2 - ATL -modules with non-zero weight. It is not known whether the inner-products on these modules are positive definite. Our construction of these modules is based on the following assumption. Let $\varphi_{(t_1, t_2)}$, $\tilde{\varphi}_{(t_1, t_2)}$ be the tangles illustrated in Figure 10. Note that $\tilde{\varphi}_{(t_2, t_1)}$ is the rotation of $\varphi_{(t_1, t_2)}$ by π . These tangles can be viewed as some sort of “rotation by one”. They have rank (t_1, t_2) . It appears that the infinite dimensional algebra $\hat{A}_k = A_2\text{-}ATL_{0,k}/A_2\text{-}ATL_{0,k}^{(k,k)}$ is generated by the two tangles $\varphi_{(k,k)}$ and $\tilde{\varphi}_{(k,k)}$, $k \geq 1$. From now on will assume that this is true.

Let $\rho_{0,k}$ be the $0, k$ -tangle given by the image of $\varphi_{(k,k)}\tilde{\varphi}_{(k,k)}$ in \hat{A}_k , illustrated in Figure 11, and let $\rho_{i,k-i}$ be the image of $\rho_{0,k}$ under the map $\varrho_k : A_2\text{-}ATL_{0,k} \rightarrow A_2\text{-}ATL_{i,k-i}$ as in Section 4.1. Then $\rho_{i,j}$ is some sort of “rotation by two”. Indeed, it can be shown for $2t_{\max} + t_{\min} = 3(i + j)$, for any $i, j \geq 0$, that $\rho_{i,j}$ is a rotation of order $i + j$ in $A_2\text{-}ATL_{i,j}/A_2\text{-}ATL_{i,j}^{(t_1, t_2)}$, i.e. $(\rho_{i,j})^{i+j} = \mathbf{1}_{i,j}$.

By drawing pictures it is easy to see that $\tilde{\varphi}_{(k,k)}\varphi_{(k,k)} = \rho_{0,k}$ in \hat{A}_k , and hence that $\varphi_{(k,k)}$, $\tilde{\varphi}_{(k,k)}$ commute in \hat{A}_k . It is also easy to check that

$$\varphi_{(k,k)}\varphi_{(k,k)}^* = \varphi_{(k,k)}^*\varphi_{(k,k)} = \tilde{\varphi}_{(k,k)}\tilde{\varphi}_{(k,k)}^* = \tilde{\varphi}_{(k,k)}^*\tilde{\varphi}_{(k,k)} = \mathbf{1}_{0,k},$$

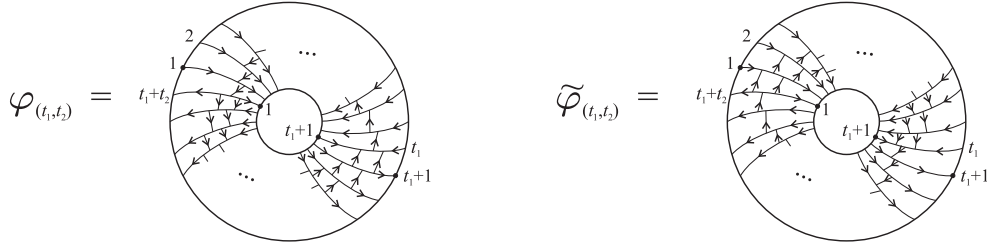


Figure 10: $\varphi_{(t_1, t_2)}$ and $\tilde{\varphi}_{(t_1, t_2)}$

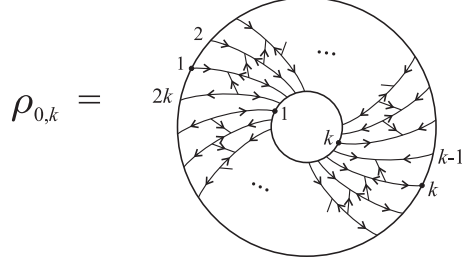


Figure 11: “Rotation” tangle $\rho_{0,k}$

so that $\varphi_{(k,k)}^* = \varphi_{(k,k)}^{-1}$ and $\tilde{\varphi}_{(k,k)}^* = \tilde{\varphi}_{(k,k)}^{-1}$ are inverse elements in \hat{A}_k . Again, by drawing pictures it is clear that $(\varphi_{(k,k)} \tilde{\varphi}_{(k,k)}^*)^k = \varphi_{(k,k)}^{2k}$. Then we have

$$\varphi_{(k,k)}^k = (\varphi_{(k,k)}^* \varphi_{(k,k)})^k \varphi_{(k,k)}^k = (\varphi_{(k,k)}^*)^k \varphi_{(k,k)}^{2k} = (\varphi_{(k,k)}^*)^k (\varphi_{(k,k)} \tilde{\varphi}_{(k,k)}^*)^k = (\tilde{\varphi}_{(k,k)}^*)^k, \quad (32)$$

and so $\tilde{\varphi}_{(k,k)}^* = \varphi_{(k,k)}^k \tilde{\varphi}_{(k,k)}^{k-1}$ and $\varphi_{(k,k)}^* = \varphi_{(k,k)}^{k-1} \tilde{\varphi}_{(k,k)}^k$.

The algebras \hat{A}_k are infinite dimensional, since $\varphi_{(k,k)}^l$, $l = 1, 2, \dots$, are all distinct and non-zero in \hat{A}_k , as are $\tilde{\varphi}_{(k,k)}^l$, $l = 1, 2, \dots$. One way to obtain a finite-dimensional A_2 - ATL -module $V^{(k,k), \gamma}$ is to consider the element $\varphi_{(k,k)} \tilde{\varphi}_{(k,k)}^*$ as acting as a scalar γ^2 in the lowest weight module $V_{0,k}^{(k,k), \gamma}$, i.e. $\varphi_{(k,k)} \tilde{\varphi}_{(k,k)}^* = \gamma^2 \mathbf{1}_{0,k}$ in $V_{0,k}^{(k,k), \gamma}$, for some $\gamma \in \mathbb{C}$. By drawing the element $\varphi_{(k,k)}^{2k}$ we see that $\varphi_{(k,k)}^{2k} = \gamma^{2k} \mathbf{1}_{0,k}$. Then we have $\varphi_{(k,k)}^* = \gamma^{-2k} \varphi_{(k,k)}^{2k-1}$, and by (32),

$$\tilde{\varphi}_{(k,k)}^k = (\varphi_{(k,k)}^*)^k = (\gamma^{-2k} \varphi_{(k,k)}^{2k-1})^k = \gamma^{-2k^2} \varphi_{(k,k)}^{2k^2-k} = \gamma^{-2k^2} \gamma^{2k(k-1)} \varphi_{(k,k)}^k = \gamma^{-2k} \varphi_{(k,k)}^k,$$

so that $\tilde{\varphi}_{(k,k)}^{2k} = \gamma^{-4k} \varphi_{(k,k)}^{2k} = \gamma^{-2k} \mathbf{1}_{0,k}$. Now $\varphi_{(k,k)}^{k+1} \tilde{\varphi}_{(k,k)}^{k-1} = \varphi_{(k,k)} \tilde{\varphi}_{(k,k)}^* = \gamma^2 \mathbf{1}_{0,k}$, so that

$$\tilde{\varphi}_{(k,k)}^{k+1} \varphi_{(k,k)}^{k+1} \tilde{\varphi}_{(k,k)}^{k-1} = \gamma^2 \tilde{\varphi}_{(k,k)}^{k+1}. \quad (33)$$

But we also have

$$\tilde{\varphi}_{(k,k)}^{k+1} \varphi_{(k,k)}^{k+1} \tilde{\varphi}_{(k,k)}^{k-1} = \varphi_{(k,k)}^{k+1} \tilde{\varphi}_{(k,k)}^{2k} = \gamma^{-2k} \varphi_{(k,k)}^{k+1}. \quad (34)$$

Comparing (33) and (34) we find that $\gamma^2 \tilde{\varphi}_{(k,k)}^{k+1} = \gamma^{-2k} \varphi_{(k,k)}^{k+1}$, which gives

$$\tilde{\varphi}_{(k,k)}^{k+1} = \gamma^{-2(k+1)} \varphi_{(k,k)}^{k+1}. \quad (35)$$

Then by (32), (35) we have

$$\tilde{\varphi}_{(k,k)} = \tilde{\varphi}_{(k,k)} \tilde{\varphi}_{(k,k)}^k (\tilde{\varphi}_{(k,k)}^*)^k = \tilde{\varphi}_{(k,k)}^{k+1} (\tilde{\varphi}_{(k,k)}^*)^k = \gamma^{-2(k+1)} \varphi_{(k,k)}^{k+1} (\gamma^{2k} (\varphi_{(k,k)}^*)^k) = \gamma^{-2} \varphi_{(k,k)}.$$

Then

$$V_{0,k}^{(k,k),\gamma} = \text{span}(\varphi_{(k,k)}^l \mid l = 0, 1, \dots, 2k-1),$$

where $\varphi_{(k,k)}^{2k} = \gamma^{2k} \mathbf{1}_{0,k}$. We see that $\varphi_{(k,k)}$ acts on $V_{0,k}^{(k,k),\gamma}$ as \mathbb{Z}_{2k} , by permuting the $2k$ basis elements $\varphi_{(k,k)}^l$, and so the A_2 - ATL -module $V_{0,k}^{(k,k),\gamma}$ decomposes as a direct sum over the $2k^{\text{th}}$ roots of unity ω of A_2 - ATL -modules $V^{(k,k),\gamma,\omega}$, where $V^{(k,k),\gamma,\omega}$ is the ω -eigenspace for the action of \mathbb{Z}_{2k} with eigenvalue ω .

For each k , we can choose a faithful trace tr' on \hat{A}_k , which we extend to a trace tr on A_2 - $ATL_{0,k}$ by $\text{tr} = \text{tr}' \circ \pi$, where π is the quotient map $\pi : A_2$ - $ATL_{0,k} \rightarrow \hat{A}_k$. We can define an inner product on A_2 - $ATL(i, j : 0, k)$ by $\langle S, T \rangle = \text{tr}(T^* S)$ for any $S, T \in A_2$ - $ATL(i, j : 0, k)$. Since $\varphi_{(k,k)}^* \varphi_{(k,k)} = \mathbf{1}_{0,k}$, the decomposition into $V^{(k,k),\gamma,\omega}$ is orthogonal. If we let $\psi_{0,k}^{\gamma,\omega}$ be the vector in $V_{0,k}^{(k,k),\gamma,\omega}$ which is proportional to $\sum_{j=0}^{2k-1} (\omega\gamma)^{-j} \varphi_{(k,k)}^j$ such that $\langle \psi_{0,k}^{\gamma,\omega}, \psi_{0,k}^{\gamma,\omega} \rangle = 1$, then $\varphi_{(k,k)} \psi_{0,k}^{\gamma,\omega} = \omega\gamma \psi_{0,k}^{\gamma,\omega}$. We see that $\dim(V_{0,k}^{(k,k),\gamma,\omega}) = 1$, and $V_{0,k}^{(k,k),\gamma,\omega}$ is the span of $\psi_{0,k}^{\gamma,\omega}$. We define the Hilbert A_2 - ATL -module $H^{(k,k),\gamma,\omega}$ to be the quotient of $V^{(k,k),\gamma,\omega}$ by the zero-length vectors with respect to this inner product.

We can also construct a finite-dimensional A_2 - ATL -module $V^{(3,0)}$ with lowest weight 2 and minimum rank $(3,0)$ as follows. Let $W_{i,j}^{(3,0)}$ be the vector space of all linear combinations of tangles with one inner disc, where the outer disc has pattern i, j , the inner disc has 3 sink vertices, with one of these vertices chosen as a distinguished vertex, and such that as we pass along the string that has this distinguished vertex as its endpoint, the region to its right must be coloured $\bar{0}$. Let $V_{i,j}^{(3,0)}$ be the quotient of $W_{i,j}^{(3,0)}$ by the ideal generated by the Kuperberg relations K1-K3. The vector space $V_{i,j}^{(3,0)}$ is infinite dimensional due to the possibility of composing the elements $\varphi_{(3,0)}$ an infinite number of times, where each $\varphi_{(3,0)}^l$, $l = 1, 2, \dots$, is a tangle which has rank $(3,0)$ and does not contain any closed circles, or embedded circles or squares. If however, we let $\varphi_{(3,0)} \tilde{\varphi}_{(3,0)}^*$ count as some scalar in $V^{(3,0)}$, i.e. $\varphi_{(3,0)} \tilde{\varphi}_{(3,0)}^* = \gamma^3 \in \mathbb{C}$, then $V_{i,j}^{(3,0)}$ is finite-dimensional since

$$\tilde{\varphi}_{(3,0)}^* = \text{[Diagram of a tangle with three strands and a central disk, with arrows indicating a specific orientation]} = \varphi_{(3,0)}^2$$

and hence $\varphi_{(3,0)}^3 = \gamma^3 \in \mathbb{C}$ (and similarly $\tilde{\varphi}_{(3,0)}^3$ is also a scalar). Since the elements $\varphi_{(3,0)}^*$ and $\varphi_{(3,0)}$ are the same, $\gamma^3 = \varphi_{(3,0)}^3 = \tilde{\varphi}_{(3,0)}^3 = \bar{\gamma}^3$, so $\gamma^3 \in \mathbb{R}$. We will denote the module $V^{(3,0)}$ where $\varphi_{(3,0)} \tilde{\varphi}_{(3,0)}^* = \gamma^3 \in \mathbb{C}$ by $V^{(3,0),\gamma}$, where $\gamma \in \mathbb{R}$.

Let $U_l, \tilde{U}_l \in A_2$ - $ATL_{i,j}$, $l = 1, \dots, j$, be the annular i, j -tangles illustrated in Figure 12. From drawing pictures, it appears that the lowest weight module $V_{0,2}^{(3,0),\gamma}$ is the span of v_l , $l = 1, \dots, 6$, where v_1 is the tangle in Figure 13, $v_{2l} = \varphi_{(2,2)} v_{2l-1}$, $l = 1, 2, 3$, and $v_{2l+1} = \tilde{U}_1 v_{2l}$, $l = 1, 2$. These are the only tangles we can find that have rank no smaller

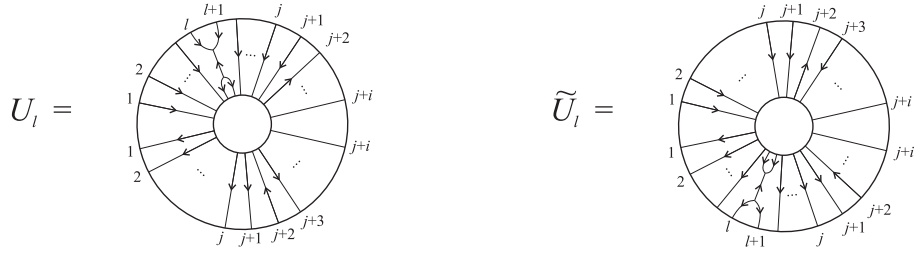


Figure 12: Annular i, j -tangles $U_l, \tilde{U}_l, l = 1, \dots, j$

than $(3, 0)$, do not contain any closed circles or embedded circles or squares, and which cannot be written as a linear combination of tangles of the form $v' \varphi_{(3,0)}^{3p}$ for some $p \in \mathbb{N}$, where v' is one of the elements v_l above, and the tangle $\varphi_{(3,0)}^{3p}$ is inserted in the inner disc of v' .

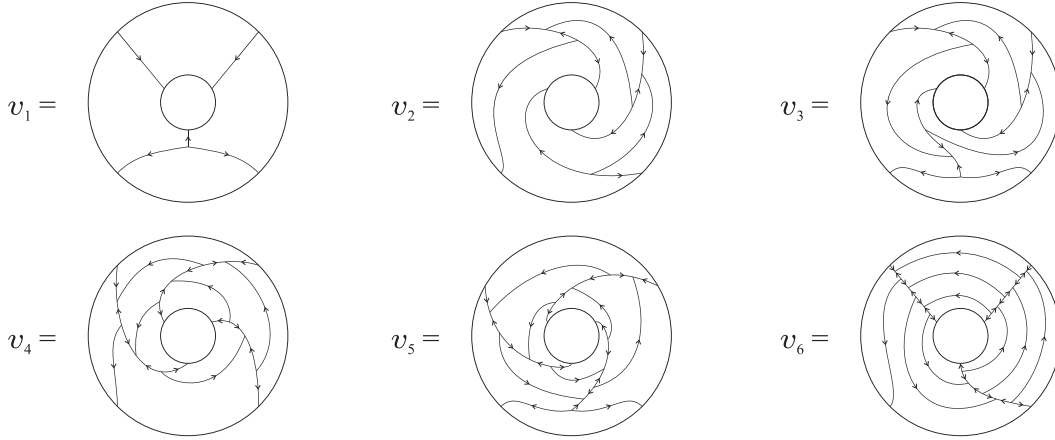


Figure 13: The basis elements $v_i, i = 1, \dots, 6$, of $V_{0,2}^{(3,0),\gamma}$

The action of $A_2\text{-}ATL_{0,2}$ on $V_{0,2}^{(3,0),\gamma}$ is given as follows. For a tangle $T \in A_2\text{-}ATL(i, j : 0, 2)$ and one of the elements v_l above, we form Tv_l and divide out by the relations K1-K3 to obtain a linear combination of tangles with pattern i, j on the outer disc and three sink vertices on the inner disc. Any tangle that has rank $< (3, 0)$ is equal to zero. For the remaining tangles, any tangle that is of the form $v' \varphi_{(3,0)}^p$ (p must necessarily be some integer multiple of 3 to respect the colouring at the inner disc) becomes $\gamma^p v' \in V_{i,j}^{(3,0),\gamma}$.

For any two elements $S, T \in V_{i,j}^{(3,0),\gamma}$, the tangle T^*S will have three (source) vertices on its outer disc and three (sink) vertices on its inner disc. We use relations K1-K3 on T^*S to obtain a linear combination $\sum_j c_j (T^*S)_j$ of tangles $(T^*S)_l$ which do not contain any closed circles, or embedded circles or squares, where $c_j \in \mathbb{C}$. We let $\langle S, T \rangle_l$ be zero if $\text{rank}((T^*S)_l) < (3, 0)$. Otherwise, $(T^*S)_l$ will be equal to $\varphi_{(3,0)}^p$ for some $p = 0, 1, 2, \dots$, and we let $\langle S, T \rangle_l$ be γ^p . We then define an inner product on $V^{(3,0),\gamma}$ by $\langle S, T \rangle = \sum_j c_j \langle S, T \rangle_j$. We define the Hilbert $A_2\text{-}ATL$ -module $H^{(3,0),\gamma}$ to be the quotient of $V^{(3,0),\gamma}$ by the zero-length vectors with respect to this inner product.

For $\gamma \neq \pm 1$, $H_{0,2}^{(3,0),\gamma}$ has dimension 6, and the action of $A_2\text{-}ATL_{0,2}$ on $H_{0,2}^{(3,0),\gamma}$ is given explicitly by

$$\begin{aligned}\varphi_{(2,2)}v_{2l-1} &= v_{2l}, & \varphi_{(2,2)}v_{2l} &= v_{2l-1}, & l &= 1, 2, 3, \\ \tilde{U}_1v_{2l-1} &= \delta v_{2l-1}, & \tilde{U}_1v_{2l} &= v_{2l+1}, & l &= 1, 2, & \tilde{U}_1v_6 &= \gamma^3v_1, \\ \tilde{\varphi}_{(2,2)}v_l &= v_l, & U_1v_l &= 0, & \text{for all } l.\end{aligned}$$

For $\gamma = \pm 1$, the dimension of $H_{0,2}^{(3,0),\pm 1}$ is 2, and $H_{0,2}^{(3,0),\pm 1}$ is the span of the elements v_1, v_2 above. The action of $A_2\text{-}ATL_{0,2}$ on $H_{0,2}^{(3,0),\pm 1}$ is given by

$$\begin{aligned}\varphi_{(2,2)}v_1 &= v_2, & \varphi_{(2,2)}v_2 &= v_1, \\ \tilde{U}_1v_1 &= \delta v_1, & \tilde{U}_1v_2 &= \gamma v_1, \\ \tilde{\varphi}_{(2,2)}v_l &= v_l, & U_1v_l &= 0, & l &= 1, 2.\end{aligned}$$

There is a similar description of modules $H^{(0,3),\gamma}$ of minimum rank $(0, 3)$, where there are now three source vertices on the inner disc. The roles of U_l and \tilde{U}_l are interchanged for $H^{(0,3),\gamma}$.

We were able to conjecture certain irreducible modules of non-zero weight that the A_2 -planar algebra $P^{\mathcal{G}}$ for the graphs $\mathcal{E}^{(8)}$ and $\mathcal{D}^{(6)}$ should contain, since the action of the rotation $\rho_{0,2}$ on the A_2 -planar algebras for these graphs was much easier to write down than for the other graphs.

For the graph $\mathcal{E}^{(8)}$, its zero-weight irreducible modules are $H^{\beta_{(0,0)}}$, $H^{\beta_{(3,0)}}$, $H^{\beta_{(0,3)}}$ and $H^{\beta_{(2,2)}}$. By computing the inner-products $\langle v_i, v_j \rangle$ of the elements $v_l \in H_{0,1}^{\beta}$ explicitly, and using Mathematica to compute the rank of the matrix $(\langle v_i, v_j \rangle)_{i,j}$, we computed the dimension of $H_{0,1}^{\beta_{(0,0)}}$, $H_{0,1}^{\beta_{(3,0)}}$, $H_{0,1}^{\beta_{(0,3)}}$ and $H_{0,1}^{\beta_{(2,2)}}$ and found that $P^{\mathcal{E}^{(8)}}$ did not contain any irreducible modules of lowest weight 1. It should be noted that Mathematica is not an open-source software, and the users have no way of knowing the reliability of results obtained using it. Similarly, by computing the dimensions of $W = H_{0,2}^{\beta_{(0,0)}} \oplus H_{0,2}^{\beta_{(3,0)}} \oplus H_{0,2}^{\beta_{(0,3)}} \oplus H_{0,2}^{\beta_{(2,2)}}$, we find that $\dim(W) = 30$ whilst $\dim(P^{\mathcal{E}^{(8)}}) = 36$, so that the dimension of $W^{\perp} \cap P^{\mathcal{E}^{(8)}}$ is 6. Then for modules of lowest weight 2, we conjecture

$$P_{0,2}^{\mathcal{E}^{(8)}} = H_{0,2}^{\beta_{(0,0)}} \oplus H_{0,2}^{\beta_{(3,0)}} \oplus H_{0,2}^{\beta_{(0,3)}} \oplus H_{0,2}^{\beta_{(2,2)}} \oplus H_{0,2}^{(3,0),\varepsilon_1} \oplus H_{0,2}^{(0,3),\varepsilon_1} \oplus H_{0,2}^{(2,2),\gamma_1,\varepsilon_2 i} \oplus H_{0,2}^{(2,2),\gamma_2,\varepsilon_3 i},$$

where $\varepsilon_i \in \{\pm 1\}$, $i = 1, 2, 3$, and $\gamma_1, \gamma_2 \in \mathbb{T}$, where the exact values of these six parameters has not yet been determined. This conjecture arises from computing the eigenvalues of the actions of $\rho_{0,2}$, U_1 and \tilde{U}_1 on $W^{\perp} \cap P_{0,2}^{\mathcal{E}^{(8)}}$. Each action is a linear transformation, which we computed by hand, and then computed using Mathematica the eigenvalues of the matrix which gives this linear transformation. These eigenvalues are

$$\rho_{0,2} : \quad 1 \text{ twice, } -1 \text{ four times,} \quad (36)$$

$$U_1, \tilde{U}_1 : \quad [4]\alpha\delta^{-2}, \text{ once, } 0 \text{ five times.} \quad (37)$$

The eigenvalues of the actions of these elements on $H_{0,2}^{(2,2),\gamma,\omega}$, $H_{0,2}^{(3,0),\gamma}$ and $H_{0,2}^{(0,3),\gamma}$ are given in the Table 1.

Then we see that $W^{\perp} \cap P_{0,2}^{\mathcal{E}^{(8)}}$ should contain one copy of both of $H_{0,2}^{(3,0),\varepsilon_1}$ and $H_{0,2}^{(0,3),\varepsilon'_1}$, $\varepsilon_1, \varepsilon'_1 \in \{\pm 1\}$, and since $P^{\mathcal{E}^{(8)}}$ is invariant under conjugation of the graph $\mathcal{E}^{(8)}$, we should

A_2 -ATL-module	Eigenvalues of the action of		
	$\rho_{0,2}$	U_1	\tilde{U}_1
$H_{0,2}^{(2,2),\gamma,\omega}$	ω^2	0	0
$H_{0,2}^{(3,0),\pm 1}$	1, -1	0 ($\times 2$)	$[4]\alpha\delta^{-2}$, 0
$H_{0,2}^{(0,3),\pm 1}$	1, -1	$[4]\alpha\delta^{-2}$, 0	0 ($\times 2$)
$H_{0,2}^{(0,3),\pm 1}, \gamma \neq \pm 1$	1 ($\times 3$), -1 ($\times 3$)	0 ($\times 6$)	$[4]\alpha\delta^{-2}$ ($\times 3$), 0 ($\times 3$)
$H_{0,2}^{(3,0),\pm 1}, \gamma \neq \pm 1$	1 ($\times 3$), -1 ($\times 3$)	$[4]\alpha\delta^{-2}$ ($\times 3$), 0 ($\times 3$)	0 ($\times 6$)

Table 1: The eigenvalues of the actions of $\rho_{0,2}$, U_1 , \tilde{U}_1 on $H_{0,2}^{(2,2),\gamma,\omega}$, $H_{0,2}^{(3,0),\gamma}$, $H_{0,2}^{(0,3),\gamma}$.

have $\varepsilon_1 = \varepsilon'_1$. Then we need to rank $(2, 2)$ modules of $H_{0,2}^{(2,2),\gamma_1,\omega}$, $H_{0,2}^{(2,2),\gamma_2,\omega}$ such that the action of $\rho_{0,2}$ on both has an eigenvalue $\omega^2 = -1$, i.e. $\omega = \pm i$. Since $P^{\mathcal{E}^{(8)}}$ is invariant under complex conjugation, we would either have $\gamma_1, \gamma_2 \in \mathbb{R}$ or else $\gamma_1 = \bar{\gamma}_2$. However, to determine the exact values of ε_i , $i = 1, 2, 3$, and γ_1, γ_2 , we would need to consider the action of $\varphi_{(2,2)}$ on $W^\perp \cap P_{0,2}^{\mathcal{E}^{(8)}}$, the computation of which would take many weeks to write down. So we have

$$P^{\mathcal{E}^{(8)}} \supset H^{\beta(0,0)} \oplus H^{\beta(3,0)} \oplus H^{\beta(0,3)} \oplus H^{\beta(2,2)} \oplus H^{(3,0),\varepsilon_1} \oplus H^{(0,3),\varepsilon_1} \oplus H^{(2,2),\gamma_1,\varepsilon_2 i} \oplus H^{(2,2),\gamma_2,\varepsilon_3 i}.$$

Similarly for the graph $\mathcal{D}^{(6)}$, we found that $P^{\mathcal{D}^{(6)}}$ also contains no irreducible modules of lowest weight 1. Computing the dimensions of $P_{0,2}^{\mathcal{D}^{(6)}}$ and $W = H_{0,2}^{\beta(0,0)} \oplus H_{0,2}^{0,\bar{0}}$ as for the $\mathcal{E}^{(8)}$ case, we find $\dim(P_{0,2}^{\mathcal{D}^{(6)}}) = 16$ and $\dim(W) = 14$. Then the dimension of $W^\perp \cap P_{0,2}^{\mathcal{D}^{(6)}}$ is 2, and hence $P_{0,2}^{\mathcal{D}^{(6)}}$ must either contain one copy of $H_{0,2}^{(3,0),\gamma}$ or else $H_{0,2}^{(2,2),\gamma_1,\omega_1} \oplus H_{0,2}^{(2,2),\gamma_2,\omega_2}$. By considering the action of $\rho_{0,2}$ on $W^\perp \cap P_{0,2}^{\mathcal{D}^{(6)}}$, we have the eigenvalue 1 twice. Then $W = H_{0,2}^{(2,2),\gamma_1,\omega_1} \oplus H_{0,2}^{(2,2),\gamma_2,\omega_2}$, where $\omega_i^2 = 1$, $i = 1, 2$. Then we see that

$$P^{\mathcal{D}^{(6)}} \supset H^{\beta(0,0)} \oplus H^{0,\bar{0}} \oplus H^{(2,2),\gamma_1,\varepsilon_1} \oplus H^{(2,2),\gamma_2,\varepsilon_2},$$

where $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$, and either $\gamma_1, \gamma_2 \in \mathbb{R}$ or else $\gamma_1 = \bar{\gamma}_2$. Again, to determine the values of ε_i, γ_i , $i = 1, 2$, explicitly requires considering the eigenvalues of the action of $\varphi_{(2,2)}$ on $W^\perp \cap P_{0,2}^{\mathcal{D}^{(6)}}$.

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